May 9, 2004

MATH 225 N2

Suggested Solutions to some of the supplementary Exercises

These exercises are only supplementary to classwork and all previous homeworks and review sheet. They do not encompass everything taught in class

- 1.1 8

Soln: $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

- 1.1 12

Soln: Inconsistent.

- 1.2 12

Soln: $\left\{ x_1 = 5 + 7x_2 - 6x_4 x_2 \text{ is free} x_3 = -3 + 2x_4 x_4 \text{ is free} \right.$

- 1.2 16

Soln:
a A unique solution
b Consistent with infinitely many solutions.

- 1.2 18

Soln: $h \neq 15$.

- 1.3 22

Soln: Construct a matrix with linearly *dependent* columns and b such that it is not spanned by them. For example take vectors from XY plane and let b be a vector in the Z axis.

- 1.4 14

Soln: No. The system has no solution.

- 1.4 18

Soln: No, the column vectors are not linearly independent. Equivalently, the equation $Ax = y$ does not have a solution for every $y \in \mathbb{R}^4$.

- 1.4 32

Soln: No. This would give only three pivot positions whereas we need 4 to span all of $\mathbb{R}^4$. In general if there are n vectors in $\mathbb{R}^m$ where $n < m$ then they cannot span all of $\mathbb{R}^m$.

- 1.5 24c

Soln: True. If zero vector is a solution then $Ax = A0 = 0$. Hence the system must be a homogeneous system.

- 1.5 24 e

Soln: False. This is true only when $Ax = b$ is a consistent system.

- 2.3 12 c.

Soln: True. If it has infinitely many solutions for any b then the row echelon form will have a row of zeros, which means one can always find b for which the system will not be consistent. Hence consistency for every b implies unique solution.

- 2.6 6

Soln: 50
45

- 4.2 26a

Soln: True. We proved this in class.

- 4.2 26 c

Soln: False. It is the set of vectors b for which $Ax = b$ has a solution.

- 5.1 26

Soln: If $A^2 = 0$. Then for any eigenvalue λ there exists an eigenvector x such that $Ax = \lambda x$. Then $A^2 x = \lambda^2 x = 0$. This implies λ^2 and hence λ is zero.

- 6.2 24 d

Soln: False. The orthogonal projection of y onto v is c^{-1} times the orthogonal projection to cv.

(1) Consider the linear system whose augmented matrix is of the form

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ -1 & 4 & 3 & 2 \\ 2 & -2 & a & 3 \end{bmatrix}$$

For what values of a will the system have a unique solution?

Soln: a = -2.

(2) Give a general solution to the following system of equations.

$$\begin{array}{rcrcrcr} 3x_1 & + & 2x_2 & - & x_3 & = & 4 \\ x_1 & - & 2x_2 & + & 2x_3 & = & 1 \\ 11x_1 & + & 2x_2 & + & x_3 & = & 14 \end{array}$$

Soln: The system has a infinitely many solution with one free variable.

(3) Let A and B be $n \times n$ matrices and let $C = AB$. Prove that if B is not invertible then C is not invertible.

Soln: If B is not invertible then $\det B = 0$. But $\det C = \det A \det B$. Therefore, $\det C = 0$, which means that C is not invertible.

(4) Let $A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}$. Solve the matrix equation $AX + B = C$ where X is a 2×2.

Soln: Compute $X = A^{-1}(C - B)$.

(6) Find all possible values of d that would make the following matrix invertible.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 9 & d \\ 1 & d & 3 \end{bmatrix}$$

Soln: For the matrix to be invertible its determinant has to be nonzero. So find values of d for which the matrix has a nonzero determinant.

(9) Let A be a $m \times n$ matrix and B is an invertible $m \times m$ matrix. Show that BA and A have the same null space and hence the same rank.

Soln: $BAx = 0$ implies $Ax = 0$ since B is invertible. Therfore, every vector in $\operatorname{Nul} BA$ is in $\operatorname{Nul} A$. Conversely, if $Ax = 0$ then multiplying by B will not change anything and we get $BAx = 0$. Therefore, $\operatorname{Nul} A$ is same as $\operatorname{Nul} BA$. Then $\operatorname{Rank} A = n - \dim \operatorname{Nul} A = n - \dim NulBA = \operatorname{Rank} BA$.

(11) Let S be a subspace of R^3 spanned by $x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Find a basis of the orthogonal complement of S and give a geometrical description of the same.

Soln: A vector $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$ is in the orthogonal complement of S if it is orthogonal to every vector in S or in fact all basis vectors of S. In this case, all vectors $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$ such that $v.x = 0$. This is same as all $a, b, c \in \mathbb{R}$ such that $a - b + c = 0$. Note this is just space $\{\begin{bmatrix} a \\ b \\ c \end{bmatrix} : a - b + c = 0\}$. So all one has to do is compute its basis.

(13) Write short answers to the following.

(i) Let $\begin{bmatrix} 1 & 3 & -1 \\ 0 & -5 & 2 \\ 2 & -1 & 0 \end{bmatrix}$ be the inverse of A. Find an appropriate matrix so that $AX = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}$. Is X invertible? Why or why not?

Soln: Since A is invertible $X = A^{-1}\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}$.Compute X. Clearly X will not be a square matrix and hence cannot be invertible.

(ii) A is a diagonalizable 2×2 matrix with eigenvalues 1 and -1. Show that $A^2 = I$.

Soln: If A is diagonalizable with eigenvalues 1 and -1. Then, $A = P\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}P^{-1}$ for some invertible matrix P. But, $A^2 = P\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}P^{-1}P\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}P^{-1} = PIP^{-1} = I$, where I is the 2×2 identity matrix.

11(iii) If A is a $n \times n$ matrix such that $AA^T = I$ then, what values will $\det A$ take?

Soln: Solve the corresponding equation for determinants and use the fact that $\det A = \det A^T$.

(iv) If $\{v_1, v_2, v_3\}$ are linearly independent vectors in $I\!R^5$ and $v_4 = v_3 - v_2 + v_1$, then is $\{v_1, v_2, v_4\}$ is linearly independent? Why or why not?

Soln: True. We already know that v_1 and v_2 are linearly independent and to get v_4 we need to use another linearly independent vector v_3. If v_4 had been dependent on v_1 and v_2 then that would imply from hte given equation that v_3 is also dependent on v_1 and v_2 which is not possible.

(v) If A has eigenvalues 1, 3 and $\frac{2}{3}$, find determinant of A.

Soln: Since A has distinct eigenvalues it is diagonalizable. Therefore, there exists invertible matrix P and diagonal matrix D such that $A = P\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{2}{3} \end{bmatrix}P^{-1}$. Then, $\det A = \det P \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{2}{3} \end{bmatrix} \det P^{-1} = \det P 2(1/\det P) = 2$.

(vi) If A is a 3×3 invertible matrix and v_1, v_2, v_3 are linearly indpendent vectors in $I\!R^3$. Show that Av_1, Av_2, Av_3 are linearly independent.

Soln: For Av_1, Av_2, Av_3 to be linearly independent the equation $c_1Av_1 + c_2Av_2 + c_3Av_3 = 0$ should have only the trivial solution. This implies $A(c_1v_1 + c_2v_2 + c_3v_3) = 0$. Since A is invertible $c_1v_1 + c_2v_2 + c_3v_3 = 0$. But v_1, v_2, v_3 are linearly independent vectors in $I\!R^3$ and hence $c_1 = c_2 = c_3 = 0$. Hence, Av_1, Av_2, Av_3 are linearly independent.

14 State True or False with justification.(*No points for just stating true or false*)

(ii) Let W be a subspace of $\mathbb{R}^4$ and v be a vector in $\mathbb{R}^4$. If $v \in W$ and $v \in W^{\perp}$ then $v = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Soln: True. If v is a vector in both W and $W^{\perp}$ then $v.v = 0$. But by properties of dot product this means that $v = 0$.

(iii) Let V be a vector space and W be a subspace of V. If $dimW = dimV$ then $W = V$.

Soln: True. If W is in V and its dimensions are the same say n. Then any n linearly independent vectors in W span W. But these are also n linearly independent vectors in V which has dimension n. Therefore they will span V as well. THis implies both the spaces are the same.