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# MATH 225 N2
## NOTES AND PROBLEMS ON DIAGONALIZATION

**Definition 0.1.** A $n \times n$ matrix $A$ is said to be diagonalizable if there is a diagonal matrix $D$ (the only nonzero entries being on the diagonal) and an invertible matrix $P$ such that $A = PDP^{-1}$.

**Theorem 0.2.** *A $n \times n$ matrix $A$ with $n$ distinct eigenvalues is diagonalizable.*

In general, if $A$ is a matrix with characteristic equation

$$\det A - \lambda I = (\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_p)^{n_p}$$

then, it will be diagonalizable when $\dim(Nul(A - \lambda_1 I)) = n_1, \cdots, \dim(Nul(A - \lambda_p I)) = n_p$.

**Example 0.3.** Consider the following transition matrix

$$A \;=\; \text{To} \begin{array}{c} \text{City} \\ \text{Suburbs} \end{array} \begin{array}{c} \text{From} \\ \begin{array}{cc} \text{City} & \text{Suburbs} \end{array} \\ \begin{bmatrix} 0.85 & 0.03 \\ 0.15 & 0.97 \end{bmatrix} \end{array}$$

The entries represent the proportions of people moving from one area to another in a given year. For example, 15 % of the people living in the city in a given year will move to the suburbs. If $p$ is the vector whose coordinates represent the number of people living in the city and suburbs in a given year, then $Ap$ represents the respective populations in the following year. To find the populations in 5 years, we have to compute $A^5 p$. Diagonalizing the matrix $A$ to compute $A^5$ will not only reduce the number of computations but also the round off errors.

**Exercise 0.4.** Diagonalize the following matrix if possible else, explain why its not diagonalizable.

$$A = \begin{bmatrix} -1 & -3 & -9 \\ 0 & 5 & 18 \\ 0 & -2 & -7 \end{bmatrix}$$

**Exercise 0.5.** Diagonalize the matrix $A$ and compute $A^{10}p$ where $A = \begin{bmatrix} 3 & -4 & 4 \\ 2 & -3 & 2 \\ 0 & 0 & -1 \end{bmatrix}$ and $p = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$.

**Exercise 0.6.** Compute $A^3$ and $A^{-1}$ where $A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

**Exercise 0.7.** If $A$ is a $3 \times 3$ invertible matrix with eigenvalues $1, 4, -2$ what are the eigenvalues of $A^{-1}$?

**Exercise 0.8.** In exercise 5.3 problem 5, state what the eigenvalues and eigenspaces are. Also, write the vector $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ as a linear combination of eigenvectors of $A$. (why can you do this? ). Now compute $A^4 v$.

**Exercise 0.9.** If $A$ has an eigenvalue 0, then can $A$ be invertible? Why or Why not?

**Exercise 0.10.** The following gives a mathematical model to describe blue-green color blindness in a given population. This gene occurs in the X-chromosome. Let $x_1^{(0)}$ be the proportion of genes for color blindness in the male population and $x_2^{(0)}$ be that in the female population. Now since male recieves one $X - chromosome$ while a female recieves on from each parent, we have the following matrix equation If $x_1^{(1)}$ and $x_2^{(1)}$ are the proportion of recessive genes in the next generation of male and female populations respectively then the following relation holds.

$$\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \end{bmatrix} = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{bmatrix}$$

Let $A$ denote the coefficient matrix and let $x^{(n)} = \begin{bmatrix} x_1^{(n)} \\ x_2^{(n)} \end{bmatrix}$ is the proportion of recessive genes in the $(n+1)^{\text{th}}$ generation of male and female populations. Diagonalise the matrix and compute what happens after $n$ generations.

**Exercise 0.11.** If $A$ is a diagonalizable matrix then find a matrix $B$ sucht that $B^2 = A$.