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Clifford modules.
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According to the authors, "the purpose of the paper is to undertake a detailed investigation of the role of Clifford algebras and spinors in the KO-theory of real vector bundles. On the one hand the use of Clifford algebras throws considerable light on the periodicity theorem for the stable orthogonal group. On the other hand the use of spinors seems essential in some of the finer points of the KO-theory which centre round the Thom isomorphism".

Part I is entirely algebraic, and studies Clifford algebras and spinor groups. The material is essentially known, but its presentation is improved. In particular, the authors emphasise the grading (over Z_2) of the Clifford algebra. They can thus write formulae which involve signs given by the standard "anticommutative law" of algebraic topology; in this way the algebra becomes simpler and more natural. A feature of this approach is that the spinor group with two pathwise-components, which is a double covering of $\mathrm{O}(k)$, arises as naturally as the spinor group with one pathwise-component, which is a double covering of $\mathrm{SO}(k)$. § 4 gives the structure of the Clifford algebras, and §§ 5, 6 discuss their representation theory.

In Part II the authors give a complete treatment of the "difference bundle construction" in K-theory. This includes a Grothendieck-type construction for the relative groups $\mathrm{K}(X,Y)$ (9.1) and a discussion of the products in these groups $(10.3, 10.4)$.

In Part III, §§ 11, 12, the authors set up the Thom isomorphism φ for real K-theory. Their construction of φ is clearly good; in particular, if φ is applied in a Whitney sum bundle, it satisfies a product formula. [Since it is one of the main objects of the paper to prove this formula, it is remarkable that the formula is not formally stated; the reader is left to deduce it from 11.3.] However, the authors seem less satisfied with their proof that φ is an isomorphism. In fact, the crucial step (11.5) amounts to case-by-case checking, using the results of R. Bott [Bull. Soc. Math. France **87** (1959), 293–310; MR0126281 (23 #A3577)], that φ has the correct effect when the base-space is a point.

An alternative construction of the Thom isomorphism has been given in R. Bott [Bull. Amer. Math. Soc. **68** (1962), 395–400; MR0152995 (27 #2966)]. This approach is convenient for computing the effect of representations, but does not lead to the product formula. It is therefore desirable to prove that the two constructions coincide. This is done in §§ 13, 14, by studying the sphere as a coset space of the spinor group.

{The reviewer remarks that the work of D. W. Anderson on KO(BG) apparently yields a third approach to the Thom isomorphism for real K-theory.}

Having shown in the course of the work that Clifford algebras are related to the Bott periodicity theorem, the paper ends (§ 15) by showing that they are related to certain questions about vector bundles over stunted projective spaces and about vector-fields on spheres.

Although the main interest of the paper lies in real K-theory, it is a feature of the method that the real and complex cases can be treated in parallel throughout.

Reviewed by *J. F. Adams*