6.5 EXERCISES

1–8 Find the average value of the function on the given interval.

1. $f(x) = 4x - x^2$, $[0, 4]$
2. $f(x) = \sin 4x$, $[-\pi, \pi]$
3. $g(x) = \sqrt[3]{x}$, $[1, 8]$
4. $g(x) = x^2\sqrt{1 + x^3}$, $[0, 2]$
5. $f(t) = te^{-t^2}$, $[0, 5]$
6. $f(\theta) = \sec^2(\theta/2)$, $[0, \pi/2]$
7. $h(x) = \cos^4 x \sin x$, $[0, \pi]$
8. $h(u) = (3 - 2u)^{-1}$, $[-1, 1]$

9–12
(a) Find the average value of f on the given interval.
(b) Find c such that $f_{\text{ave}} = f(c)$.
(c) Sketch the graph of f and a rectangle whose area is the same as the area under the graph of f.

9. $f(x) = (x - 3)^2$, $[2, 5]$
10. $f(x) = \sqrt{x}$, $[0, 4]$
11. $f(x) = 2\sin x - \sin 2x$, $[0, \pi]$
12. $f(x) = 2x/(1 + x^2)^2$, $[0, 2]$

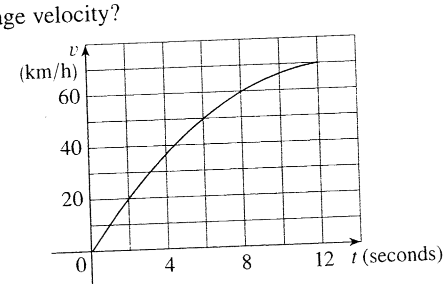
13. If f is continuous and $\int_1^3 f(x)\,dx = 8$, show that f takes on the value 4 at least once on the interval $[1, 3]$.

14. Find the numbers b such that the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal to 3.

15. The table gives values of a continuous function. Use the Midpoint Rule to estimate the average value of f on $[20, 50]$.

x	20	25	30	35	40	45	50
$f(x)$	42	38	31	29	35	48	60

16. The velocity graph of an accelerating car is shown.
(a) Estimate the average velocity of the car during the first 12 seconds.
(b) At what time was the instantaneous velocity equal to the average velocity?



17. In a certain city the temperature (in °F) t hours after 9 AM was modeled by the function

$$T(t) = 50 + 14\sin\frac{\pi t}{12}$$

Find the average temperature during the period from 9 AM to 9 PM.

18. (a) A cup of coffee has temperature 95°C and takes 30 minutes to cool to 61°C in a room with temperature 20°C. Use Newton's Law of Cooling (Section 3.8) to show that the temperature of the coffee after t minutes is

$$T(t) = 20 + 75e^{-kt}$$

where $k \approx 0.02$.
(b) What is the average temperature of the coffee during the first half hour?

19. The linear density in a rod 8 m long is $12/\sqrt{x + 1}$ kg/m, where x is measured in meters from one end of the rod. Find the average density of the rod.

20. If a freely falling body starts from rest, then its displacement is given by $s = \frac{1}{2}gt^2$. Let the velocity after a time T be v_T. Show that if we compute the average of the velocities with respect to t we get $v_{\text{ave}} = \frac{1}{2}v_T$, but if we compute the average of the velocities with respect to s we get $v_{\text{ave}} = \frac{2}{3}v_T$.

21. Use the result of Exercise 79 in Section 5.5 to compute the average volume of inhaled air in the lungs in one respiratory cycle.

22. The velocity v of blood that flows in a blood vessel with radius R and length l at a distance r from the central axis is

$$v(r) = \frac{P}{4\eta l}(R^2 - r^2)$$

where P is the pressure difference between the ends of the vessel and η is the viscosity of the blood (see Example 7 in Section 3.7). Find the average velocity (with respect to r) over the interval $0 \leqslant r \leqslant R$. Compare the average velocity with the maximum velocity.

23. Prove the Mean Value Theorem for Integrals by applying the Mean Value Theorem for derivatives (see Section 4.2) to the function $F(x) = \int_a^x f(t)\,dt$.

24. If $f_{\text{ave}}[a, b]$ denotes the average value of f on the interval $[a, b]$ and $a < c < b$, show that

$$f_{\text{ave}}[a, b] = \frac{c - a}{b - a} f_{\text{ave}}[a, c] + \frac{b - c}{b - a} f_{\text{ave}}[c, b]$$

■ Figure 8 shows the interpretation of the arc length function in Example 4. Figure 9 shows the graph of this arc length function. Why is $s(x)$ negative when x is less than 1?

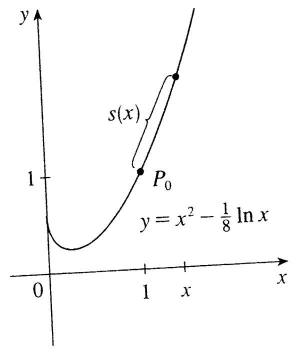


FIGURE 8

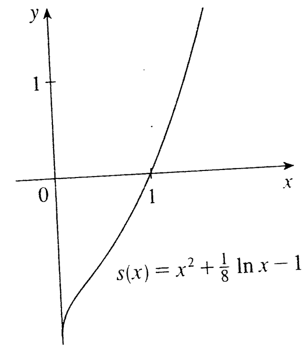


FIGURE 9

8.1 EXERCISES

1. Use the arc length formula (3) to find the length of the curve $y = 2x - 5$, $-1 \leqslant x \leqslant 3$. Check your answer by noting that the curve is a line segment and calculating its length by the distance formula.

2. Use the arc length formula to find the length of the curve $y = \sqrt{2 - x^2}$, $0 \leqslant x \leqslant 1$. Check your answer by noting that the curve is part of a circle.

3–6 Set up, but do not evaluate, an integral for the length of the ~~curve~~.

3. $y = \cos x, \quad 0 \leqslant x \leqslant 2\pi$

4. $y = xe^{-x^2}, \quad 0 \leqslant x \leqslant 1$

5. $x = y + y^3, \quad 1 \leqslant y \leqslant 4$

6. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

7–18 Find the length of the curve.

$\boxed{7.}$ $y = 1 + 6x^{3/2}, \quad 0 \leqslant x \leqslant 1$

8. $y^2 = 4(x + 4)^3, \quad 0 \leqslant x \leqslant 2, \quad y > 0$

9. $y = \dfrac{x^5}{6} + \dfrac{1}{10x^3}, \quad 1 \leqslant x \leqslant 2$

10. $x = \dfrac{y^4}{8} + \dfrac{1}{4y^2}, \quad 1 \leqslant y \leqslant 2$

$\boxed{11.}$ $x = \frac{1}{3}\sqrt{y}\,(y - 3), \quad 1 \leqslant y \leqslant 9$

12. $y = \ln(\cos x), \quad 0 \leqslant x \leqslant \pi/3$

$\boxed{13.}$ $y = \ln(\sec x), \quad 0 \leqslant x \leqslant \pi/4$

14. $y = 3 + \frac{1}{2}\cosh 2x, \quad 0 \leqslant x \leqslant 1$

15. $y = \ln(1 - x^2), \quad 0 \leqslant x \leqslant \frac{1}{2}$

16. $y = \sqrt{x - x^2} + \sin^{-1}(\sqrt{x})$

17. $y = e^x, \quad 0 \leqslant x \leqslant 1$

18. $y = \ln\left(\dfrac{e^x + 1}{e^x - 1}\right), \quad a \leqslant x \leqslant b, \quad a > 0$

19–20 Find the length of the arc of the curve from point P to point Q.

19. $y = \frac{1}{2}x^2, \quad P(-1, \frac{1}{2}), \quad Q(1, \frac{1}{2})$

20. $x^2 = (y - 4)^3, \quad P(1, 5), \quad Q(8, 8)$

21–22 Graph the curve and visually estimate its length. Then find its exact length.

21. $y = \frac{2}{3}(x^2 - 1)^{3/2}, \quad 1 \leqslant x \leqslant 3$

22. $y = \dfrac{x^3}{6} + \dfrac{1}{2x}, \quad \frac{1}{2} \leqslant x \leqslant 1$

23–26 Use Simpson's Rule with $n = 10$ to estimate the arc length of the curve. Compare your answer with the value of the integral produced by your calculator.

23. $y = xe^{-x}, \quad 0 \leqslant x \leqslant 5$

24. $x = y + \sqrt{y}, \quad 1 \leqslant y \leqslant 2$

25. $y = \sec x, \quad 0 \leqslant x \leqslant \pi/3$

26. $y = x \ln x, \quad 1 \leqslant x \leqslant 3$

27. (a) Graph the curve $y = x\sqrt[3]{4 - x}$, $0 \leqslant x \leqslant 4$.
(b) Compute the lengths of inscribed polygons with $n = 1, 2$, and 4 sides. (Divide the interval into equal subintervals.) Illustrate by sketching these polygons (as in Figure 6).
(c) Set up an integral for the length of the curve.
(d) Use your calculator to find the length of the curve to four decimal places. Compare with the approximations in part (b).

28. Repeat Exercise 27 for the curve

$$y = x + \sin x \qquad 0 \leqslant x \leqslant 2\pi$$

29. Use either a computer algebra system or a table of integrals to find the *exact* length of the arc of the curve $y = \ln x$ that lies between the points $(1, 0)$ and $(2, \ln 2)$.

30. Use either a computer algebra system or a table of integrals to find the *exact* length of the arc of the curve $y = x^{4/3}$ that lies between the points $(0, 0)$ and $(1, 1)$. If your CAS has trouble evaluating the integral, make a substitution that changes the integral into one that the CAS can evaluate.

31. Sketch the curve with equation $x^{2/3} + y^{2/3} = 1$ and use symmetry to find its length.

32. (a) Sketch the curve $y^3 = x^2$.
(b) Use Formulas 3 and 4 to set up two integrals for the arc length from $(0, 0)$ to $(1, 1)$. Observe that one of these is an improper integral and evaluate both of them.
(c) Find the length of the arc of this curve from $(-1, 1)$ to $(8, 4)$.

33. Find the arc length function for the curve $y = 2x^{3/2}$ with starting point $P_0(1, 2)$.

34. (a) Graph the curve $y = \frac{1}{3}x^3 + 1/(4x)$, $x > 0$.
(b) Find the arc length function for this curve with starting point $P_0(1, \frac{7}{12})$.
(c) Graph the arc length function.

35. Find the arc length function for the curve $y = \sin^{-1}x + \sqrt{1 - x^2}$ with starting point $(0, 1)$.

36. A steady wind blows a kite due west. The kite's height above ground from horizontal position $x = 0$ to $x = 80$ ft is given by $y = 150 - \frac{1}{40}(x - 50)^2$. Find the distance traveled by the kite.

37. A hawk flying at 15 m/s at an altitude of 180 m accidentally drops its prey. The parabolic trajectory of the falling prey is described by the equation

$$y = 180 - \frac{x^2}{45}$$

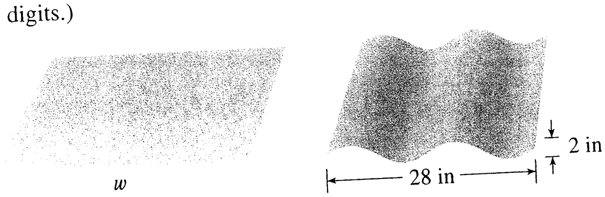
until it hits the ground, where y is its height above the ground and x is the horizontal distance traveled in meters. Calculate the distance traveled by the prey from the time it is dropped until the time it hits the ground. Express your answer correct to the nearest tenth of a meter.

38. The Gateway Arch in St. Louis (see the photo on page 256) was constructed using the equation

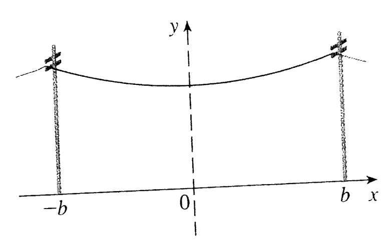
$$y = 211.49 - 20.96 \cosh 0.03291765x$$

for the central curve of the arch, where x and y are measured in meters and $|x| \leqslant 91.20$. Set up an integral for the length of the arch and use your calculator to estimate the length correct to the nearest meter.

39. A manufacturer of corrugated metal roofing wants to produce panels that are 28 in. wide and 2 in. thick by processing flat sheets of metal as shown in the figure. The profile of the roofing takes the shape of a sine wave. Verify that the sine curve has equation $y = \sin(\pi x/7)$ and find the width w of a flat metal sheet that is needed to make a 28-inch panel. (Use your calculator to evaluate the integral correct to four significant digits.)



40. (a) The figure shows a telephone wire hanging between two poles at $x = -b$ and $x = b$. It takes the shape of a catenary with equation $y = c + a\cosh(x/a)$. Find the length of the wire.
(b) Suppose two telephone poles are 50 ft apart and the length of the wire between the poles is 51 ft. If the lowest point of the wire must be 20 ft above the ground, how high up on each pole should the wire be attached?



41. Find the length of the curve

$$y = \int_1^x \sqrt{t^3 - 1}\, dt \qquad 1 \leqslant x \leqslant 4$$

42. The curves with equations $x^n + y^n = 1$, $n = 4, 6, 8, \ldots$, are called **fat circles**. Graph the curves with $n = 2, 4, 6, 8$, and 10 to see why. Set up an integral for the length L_{2k} of the fat circle with $n = 2k$. Without attempting to evaluate this integral, state the value of $\lim_{k\to\infty} L_{2k}$.