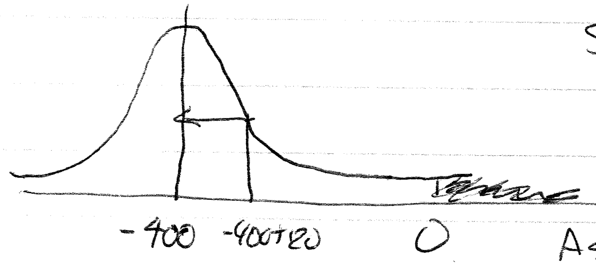
Math 461
HW 8 Sols
Due 11/16/09

1. As we've seen before, $E(x) = 17(.1)+(-3)(.9) = 1.7-2.7 = -1$ and $E(x^2) = 17^2(.1)+(-3)^2(.9) = 28.9+8.1 = 37$, so $Var(x) = 37-(-1)^2 = 36$ and $SD(x) = \sqrt{36} = 6$.

We already know that $E(S) = 400E(x) = -400$ and $Var(S) = 400 Var(x) = 400 \cdot 36 = 14400$, so $SD(x) = 120$, so in the normalized terms,



$$S \geq 0 \iff Z \geq \frac{0-(-400)}{120} = \frac{10}{3}$$

$$P(S \geq 0) = 1 - \Phi\left(\frac{10}{3}\right) \simeq .000429$$

As we saw before, Mathematica gave the exact answer of .00105. Is this close?

Depends! It's off by more than a factor of two, but only by .00062. You decide if that is a good estimate.

2. This corresponds precisely to a random variable for which $P(X=1)=.3$, $P(X=3)=.5$, $P(X=6)=.2$, so $E(x) = (.3)1+.5(3)+(.2)6 = 3$ and $E(x^2) = (.3)1^2+(.5)3^2+(.2)6^2 = .3+4.5+7.2 = 12$, so $Var(x) = E(x^2)-(E(x))^2 = 12-3^2 = 3$

Or: use Mathematica to see that

$$.3e^t + .5e^{3t} + .2e^{6t} = 1 + 3t + \frac{12}{2!}t^2 + \frac{57}{3!}t^3 + \frac{300}{4!}t^4 + \cdots$$

so $E(x)=3$, $E(x^2)=12$, $E(x^3)=57$, $E(x^4)=300$, etc.

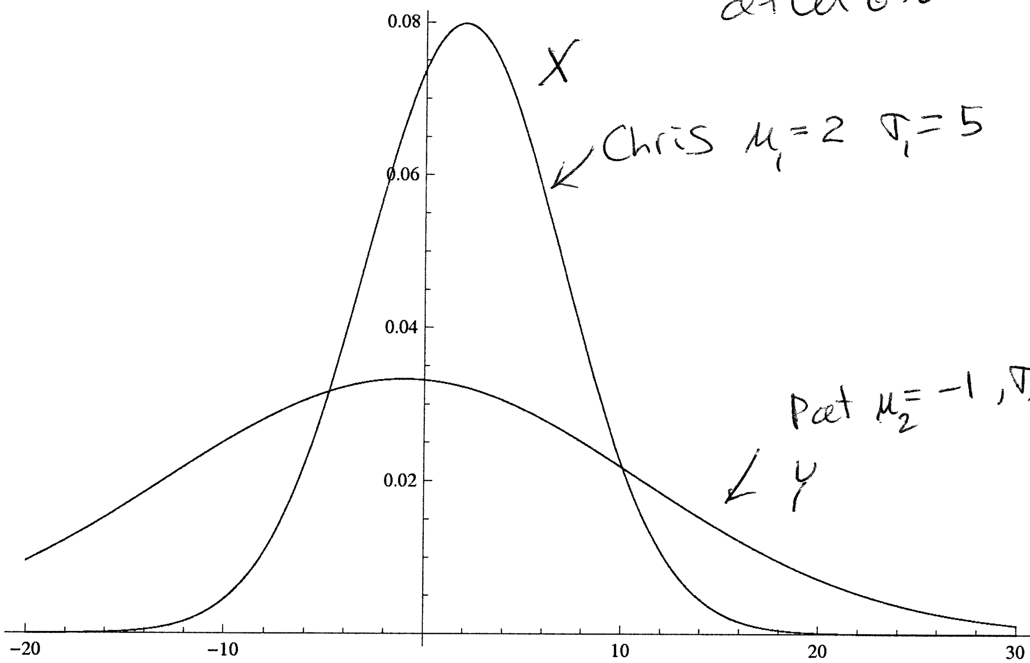
3. If you play the game N times, you have an expected return of $(-.1)N$ with variance $= N \cdot 2^2 = 4N$. For this reason, using the approximation from the Central Limit Theorem,

$$P(X \geq 0) = P\left(Z \geq \frac{0-(-.1N)}{\sqrt{4N}}\right) = P\left(Z \geq \frac{.1N}{2\sqrt{N}}\right)$$

$$= P\left(Z \geq \frac{\sqrt{N}}{20}\right) = 1 - \Phi\left(\frac{\sqrt{N}}{20}\right).$$ So if you want to win with prob. .02, this makes $\Phi\left(\frac{\sqrt{N}}{20}\right) \approx .98$, so $\frac{\sqrt{N}}{20} \approx 2.05$ (from the data given)

$\sqrt{N} \simeq 41$ or $N \approx 41^2 = 1681$

4. Exact graphs of Pat's and Chris's arrival, in terms of the pdf and in units of minutes after 6:00



(a) Normalized Chris $P\left(\frac{0-2}{5} \le Z \le \frac{10-2}{5}\right) = \Phi\left(\frac{8}{5}\right) - \Phi\left(-\frac{2}{5}\right)$

Normalized Pat $P\left(\frac{0-(-1)}{12} \le Z \le \frac{10-(-1)}{12}\right) = \Phi\left(\frac{11}{12}\right) - \Phi\left(\frac{1}{12}\right)$

$P(6{:}00 \le \text{Pat arrives} \le 6{:}10) = \Phi\left(\frac{11}{12}\right) - \Phi\left(\frac{1}{12}\right) \approx .8203 - .5332 \approx .2871$

$P(6{:}00 \le \text{Chris arrives} \le 6{:}10) = \Phi\left(\frac{8}{5}\right) - \Phi\left(\frac{-2}{5}\right) \approx .9452 - .3446 \approx .6006$

The probability for both is the product.

$\left(\Phi\left(\frac{8}{5}\right) - \Phi\left(\frac{-2}{5}\right)\right)\left(\Phi\left(\frac{11}{12}\right) - \Phi\left(\frac{1}{12}\right)\right) \approx .1725$

Note that $P(\text{Chris} \ge 6{:}10) = 1 - .9452 = .0548$

$P(\text{Pat} \ge 6{:}10) = 1 - .8203 \approx .1797$

$N(\text{Chris} \le 6{:}00) = .3446$

$P(\text{Pat} \le 6{:}00) = .5332$

(b) Look at $Y - X$. Pat arrives before Chris if $Y \le X$

or, if $Y - X \le 0$. $E(Y - X) = -1 - 2 = -3$

$\text{Var}(Y-X) = \text{Var}(Y) + (-1)^2\,\text{Var}(X) = \sigma_2^2 + \sigma_1^2 = 144 + 25 = 169 = 13^2$

$P(Y - X \le 0) = P\left(Z \le \frac{0-(-3)}{13}\right) = \Phi\left(\frac{3}{13}\right) \approx .591$