Math 461
Class notes
11/20/09

1. Let's get the binomial identity out of the way first.

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{\infty} \binom{n}{k} x^k \quad \text{because } \binom{n}{k}=0 \text{ when } k>n.$$

$$(1+x)^m = \sum_{\ell=0}^{m} \binom{m}{\ell} x^\ell = \sum_{\ell=0}^{\infty} \binom{m}{\ell} x^\ell \quad \text{(same reason)}$$

$$\sum_{r=0}^{\infty} \binom{m+n}{r} x^r = (1+x)^n (1+x)^m = \sum_{k=0}^{\infty} \binom{n}{k} x^k \sum_{\ell=0}^{\infty} \binom{m}{\ell} x^\ell$$

$$= \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \underbrace{\binom{n}{k}\binom{m}{\ell} x^{k+\ell}}$$

so $\binom{m+n}{r}$ is the coefficient of x^r in here, which happens when $k+\ell=r$

$$\binom{m+n}{r} = \sum_{k+\ell=r} \binom{n}{k}\binom{m}{\ell} = \sum_{k} \binom{n}{k}\binom{m}{r-k}.$$

The combinatorial way to think about it is that if you are selecting r objects from a set of size $m+n$, you can split this into two sets, of size m and size n respectively, choose k from the first set, $r-k$ from the second set, and sum over all choices of k

2. A <u>deck of cards</u> is a cartesian product $\{(\text{value}, \text{suit})\}$, where value $\in \{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K\}$ ← 13 values
suit $\in$ {Hearts, Clubs, Spades, Diamonds} ← 4 suits

There are $13 \times 4 = 52$ cards.

A <u>k-card hand</u> is an unordered subset of k different cards.

There are $\binom{52}{k}$ different k-card hands.

There are $\binom{52}{5}$ 5-card hands.

If a hand has k hearts and $5-k$ <u>non</u>-hearts, then it can be put together in $\binom{13}{k}\binom{39}{5-k}$ ways, $0 \le k \le 5$

The probability a 5-card hand has no hearts is

$$\frac{\binom{39}{5}}{\binom{52}{5}}$$

The probability a 5 card hand has ≥ 1 hearts is either

$$\frac{\binom{13}{1}\binom{39}{4}+\binom{13}{2}\binom{39}{3}+\binom{13}{3}\binom{39}{2}+\binom{13}{4}\binom{39}{1}+\binom{13}{5}\binom{39}{0}}{\binom{52}{5}}$$

counting 1, 2, 3, 4, 5 hearts respectively,

or

$$1-\frac{\binom{39}{5}}{\binom{52}{5}}=\frac{\binom{52}{5}-\binom{13}{0}\binom{39}{5}}{\binom{52}{5}}$$ because $P(A) = 1-P(A^c)$.

The equality of these two expressions uses the binomial identity on 1.

The probability that a 5 card hand has no aces is

$$\frac{\binom{4}{0}\binom{48}{5}}{\binom{52}{5}}=\frac{\binom{48}{5}}{\binom{52}{5}}$$: the 5 cards are chosen from the 48 non-aces

The probability that a 5 card hand has no hearts and no aces is $\frac{\binom{36}{5}}{\binom{52}{5}}$ because there are 13 hearts, 4 aces and 1 Ace of hearts: $16 = 13+4-1$.

Note. $P(0 \text{ Aces} \mid 0 \text{ hearts}) = \dfrac{P(0 \text{ Aces} + 0 \text{ hearts})}{P(0 \text{ hearts})} = \dfrac{\binom{36}{5}/\binom{52}{5}}{\binom{39}{5}/\binom{52}{5}} \to \dfrac{.145055}{.221534} \approx .654776$

$$P(0 \text{ Aces}) = \frac{\binom{48}{5}}{\binom{52}{5}} \approx .658842$$

These are close, but not equal.