A new bound for Pólya’s Theorem with applications to polynomials positive on polyhedra

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1 Introduction

Fix a positive integer $n$ and let $\mathbb{R}[X] := \mathbb{R}[x_1, \ldots, x_n]$. We write $\Delta_n$ for the simplex $\{(x_1, \ldots, x_n) \mid x_i \geq 0, \sum_i x_i = 1\}.$

Pólya’s Theorem ([6], [4, pp.57-59]) says that if $f \in \mathbb{R}[X]$ is homogeneous and positive on $\Delta_n$, then for sufficiently large $N$ all the coefficients of

$$(x_1 + \cdots + x_n)^N f(x_1, \ldots, x_n)$$

are positive. In this note, we give an explicit bound for $N$ and give an application to some special representations of polynomials positive on polyhedra. In particular, we give a bound for the degree of a representation of a polynomial positive on a convex polyhedron as a positive linear combination of products of the linear polynomials defining the polyhedron.

We use the following multinomial notation: For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, let $X^\alpha$ denote $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and write $|\alpha|$ for $\alpha_1 + \cdots + \alpha_n$. If $|\alpha| = d$, define $c(\alpha) := \frac{d}{\alpha_1! \cdots \alpha_n!}$. Let us fix homogeneous $f \in \mathbb{R}[X]$ of degree $d$,

$$f(X) = \sum_{|\alpha|=d} a_\alpha X^\alpha = \sum_{|\alpha|=d} c(\alpha) b_\alpha X^\alpha,$$

and let $L = L(f) := \max_{|\alpha|=d} |b_\alpha|$ and $\lambda = \lambda(f) := \min_{X \in \Delta_n} f(X)$.

Our main theorem is:

**Theorem 1.** Suppose that $f \in \mathbb{R}[X]$ is a form as above. If

$$N > \frac{d(d-1) L}{2} \lambda - d,$$

then $(x_1 + \cdots + x_n)^N f(x_1, \ldots, x_n)$ has positive coefficients.
Note in particular that the bound does not depend on $n$, the number of variables. This bound improves (by a factor of roughly $4n$) the bound in the paper [1], which in any case contains an error in the proof, see [2]. In [7, Ex. 3.5], we considered a special case equivalent to $f(x, y) = x^2 - (2 - \delta)xy + y^2$, for which $L = \min\{1, 1 - \frac{2 - \delta}{2}\} = 1$ and
\[
\lambda = \min_{0 < t < 1} (t - (1 - t))^2 + \delta t(1 - t) = \min_{0 < t < 1} 1 - (4 - \delta)t(1 - t) = \frac{\delta}{4};
\]
thus Theorem 1 gives $N > \frac{\delta}{\delta - 2}$. In fact, $(x + y)^N f(x, y)$ has positive coefficients precisely when $N \geq 2\lceil \frac{\delta}{\delta - 2} \rceil - 3$, so Theorem 1 is sharp.

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2 The bound

In this section, we prove Theorem 1. We begin with some notation. For a positive number $t$, a non-negative integer $m$, and a single real variable $x$, define
\[
(x)_t^m := x(x - t) \cdots (x - (m - 1)t) = \prod_{i=0}^{m-1} (x - it).
\]
Note for later reference that
\[
(ty)^d_i = \prod_{i=0}^{d-1} (ty - (i - 1)t) = t^d(y)_1^d,
\]
and if $m > n$ are both integers, then $(n)_m^m = 0$, since one of the factors in the definition is zero. It follows immediately that in the special case where $x = k/M$ and $t = 1/M$, where $M$ is a positive integer, we have
\[
\left( \frac{k}{M} \right)_{1/M}^m = \frac{1}{M^m} \prod_{i=0}^{m-1} (k - i) = \begin{cases} \frac{1}{M^m} \frac{k!}{(k - m)!} = \frac{m!}{M^m} \binom{k}{m}, & \text{if } m \leq k; \\ 0, & \text{otherwise}. \end{cases}
\]

We fix $f = \sum a_\alpha x^\alpha$ and suppose that $f > 0$ on $\Delta_n$. We assume throughout that $d = \deg f > 1$; the $d = 1$ case is trivial. Following Pólya, we make the explicit computation:
\[
(x_1 + \cdots + x_n)^N f(x_1, \ldots, x_n) = \sum_{|\beta| = N} \frac{N!}{\beta_1! \cdots \beta_n!} x_1^{\beta_1} \cdots x_n^{\beta_n} \times \sum_{|\alpha| = d} a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}.
\]

Write $\alpha \preceq \beta$ if $\alpha_i \leq \beta_i$ for $1 \leq i \leq n$. For $|\beta| = N + d$, denote the coefficient of $x_1^{\beta_1} \cdots x_n^{\beta_n}$ in $(x_1 + \cdots + x_n)^N f(x_1, \ldots, x_n)$ by $A_\beta$. Then
\[
A_\beta = \sum_{|\alpha| = d, \alpha \preceq \beta} \frac{N!}{(\beta_1 - \alpha_1)! \cdots (\beta_n - \alpha_n)!} \cdot a_\alpha
\]
\[
= \frac{N!(N + d)^d}{\beta_1! \cdots \beta_n!} \sum_{|\alpha| = d, \alpha \preceq \beta} a_\alpha \prod_{t=1}^n \frac{\beta_t!}{(\beta_t - \alpha_t)! (N + d)^{\alpha_t}}.
\]
We now express $A_\beta$ using the $|x|^a$ notation and (2):

$$A_\beta = \frac{N!(N+d)^d}{\beta_1! \cdots \beta_n!} \sum_{|\alpha|=d} a_\alpha \left( \frac{\beta_1}{N+d} \right)^{\alpha_1} \cdots \left( \frac{\beta_n}{N+d} \right)^{\alpha_n}. \quad (3)$$

(If $\alpha \nleq \beta$, then the extra terms added in (3) are just 0.) Still following Pólya, define

$$f_t(x_1, \ldots, x_n) := \sum_{|\alpha|=d} a_\alpha (x_1)^{\alpha_1} \cdots (x_n)^{\alpha_n}.$$ 

Clearly, $f_t \to f$ uniformly on $\Delta_n$, so that for $t$ sufficiently small, $f_t$ is also positive on $\Delta_n$. In view of the foregoing, this means that for $N$ sufficiently large, and all $\beta$ with $|\beta| = N + d$ (so that $f_t(N+d)^{-1}$ is evaluated on $\Delta_n$),

$$A_\beta = \frac{N!(N+d)^d}{\beta_1! \cdots \beta_n!} f_t(N+d)^{-1}(\frac{\beta_1}{N+d}, \ldots, \frac{\beta_n}{N+d}) > 0. \quad (4)$$

It follows that all the coefficients of $(x_1 + \cdots + x_n)^N f(x_1, \ldots, x_n)$ are positive.

We now extend Pólya’s work. Let us drop the constant factor in (4) and set $t = \frac{1}{N+d}$, $y_k = \frac{\beta_k}{N+d}$, and keep in mind that $\sum_k y_k = 1$. We have

$$f_t(y_1, \ldots, y_n) = f(y_1, \ldots, y_n) - \sum_{|\alpha|=d} a_\alpha (y_1^{\alpha_1} \cdots y_n^{\alpha_n} - (y_1)^{\alpha_1} \cdots (y_n)^{\alpha_n}).$$

Using the information about $f$, we see that

$$f_t(y_1, \ldots, y_n) \geq \lambda - L \sum_{|\alpha|=d} \frac{d!}{\alpha_1! \cdots \alpha_n!} |y_1^{\alpha_1} \cdots y_n^{\alpha_n} - (y_1)^{\alpha_1} \cdots (y_n)^{\alpha_n}|. \quad (5)$$

If $\alpha_k > \beta_k$, then $(y_k)^{\alpha_k} = 0$, so $y_k^{\alpha_k} \geq (y_k)^{\alpha_k} \geq 0$ for all $k$; hence we may drop the absolute value in (5).

By the Multinomial Theorem,

$$\sum_{|\alpha|=d} \frac{d!}{\alpha_1! \cdots \alpha_n!} y_1^{\alpha_1} \cdots y_n^{\alpha_n} = (y_1 + \cdots + y_n)^d = 1,$$

and by the iterated Vandermonde-Chu identity (see below for a proof),

$$\sum_{|\alpha|=d} \frac{d!}{\alpha_1! \cdots \alpha_n!} (y_1)^{\alpha_1} \cdots (y_n)^{\alpha_n} = (y_1 + \cdots + y_n)^d = \prod_{k=0}^{d-1} (1 - kt). \quad (6)$$

Thus by (5), we are done if we can show that

$$\lambda - L (1 - (1-t) \cdots (1-(d-1)t)) \geq 0. \quad (7)$$

Suppose now that

$$t = \frac{1}{N+d} < \frac{2}{d(d-1)\lambda}.$$ 

It is easy to prove by induction that if $0 \leq w_j \leq 1$, then $\prod (1-w_j) \geq 1 - \sum w_j$. Since $\lambda \leq f(1,0,\ldots,0) \leq L$ and $d \geq 2$, we have $t < \frac{\lambda}{d-1}$, hence

$$(1 - (1-t) \cdots (1-(d-1)t)) < t(1+2+\cdots+(d-1)) = t\frac{(d-1)d}{2} < \frac{\lambda}{L},$$
and we are done.

What remains is to prove the iterated Vandermonde-Chu identity (reference thanks to Doron Zeilberger):

$$\sum_{|\alpha|=d} \frac{d!}{\alpha_1! \cdots \alpha_n!} (X_1)^{\alpha_1} \cdots (X_n)^{\alpha_n} = (X_1 + \cdots + X_n)^d. \tag{8}$$

We first prove (8) combinatorially in the special case that $t = 1$ and $X_k = y_k$ is a non-negative integer:

$$\sum_{|\alpha|=d} \frac{d!}{\alpha_1! \cdots \alpha_n!} (y_1)^{\alpha_1} \cdots (y_n)^{\alpha_n} = (y_1 + \cdots + y_n)^d. \tag{9}$$

Consider $n$ sets $S_1, \ldots, S_n$ of distinct elements, where $|S_k| = y_k$, and let $S = \cup S_k$. Then $|S| = \sum y_k := y$, and the number of $d$-tuples of distinct elements from $S$ is plainly $y(y - 1) \cdots (y - (d - 1))$, which is the right-hand side of (9). We now count the number of $d$-tuples in a different way. For each $n$-tuple $\alpha$ with $|\alpha| = d$, consider the number of such $d$-tuples in which there are $\alpha_k$ distinct elements from $S_k$, $1 \leq k \leq n$. There are $\binom{y_k}{\alpha_k}$ ways to choose these elements, and $d!$ ways to arrange them, and so, altogether,

$$d! \prod_{k=1}^{n} \binom{y_k}{\alpha_k} = \frac{d!}{\alpha_1! \cdots \alpha_n!} (y_1)^{\alpha_1} \cdots (y_n)^{\alpha_n}$$

d-tuples. We sum over all possible choices of $\alpha$ to get the left-hand side of (9), completing the proof in the special case that the $y_k$’s are non-negative integers. But both sides of (9) are polynomials in the $y_k$’s, and their difference is a polynomial which vanishes on $\mathbb{N}^n$. It’s easy to see that such a polynomial must vanish identically, and so (9) is in fact an identity for all real $y_k$. Finally, let $y_k = X_k/t$ in (9) and multiply through by $t^d$, keeping (1) in mind. Then we have proved (8).

### 3 Polynomials positive on polyhedra

Suppose $P \subseteq \mathbb{R}^n$ is a convex polyhedron with non-empty interior, bounded by linear polynomials $\lambda_1, \ldots, \lambda_k \in \mathbb{R}[X]$. We always choose the sign of the $\lambda_i$’s so that $P = \{X \mid \lambda_i(X) \geq 0 \text{ for all } i\}$. Then it is an remarkable fact that any polynomial which is strictly positive on $P$ can be written as a positive linear combination of powers of the $\lambda_i$’s:

**Theorem 2.** Given $P$ as above and suppose $f \in \mathbb{R}[X]$ is strictly positive on $P$. Then for some $m \in \mathbb{N}$, $f$ has a representation

$$f = \sum_{|\alpha| \leq m} b_\alpha \lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n}, \tag{10}$$

where $b_\alpha \geq 0$ for all $\alpha$.

Theorem 2 was first proven by Handelman [3]. His proof is non-constructive; it uses a representation theorem similar to the Kadison-Dubois Theorem. In this section, using our bound for Pólya’s Theorem, we give an upper bound for the degree $m$ of a representation (10).

We begin with the case where $P$ is a simplex. In this case, a degree bound for the representation follows almost immediately from the bound in Pólya’s Theorem. Let $S$ be an $n$-simplex...
in $\mathbb{R}^n$, with vertices $\{v_0, \ldots, v_n\}$ and let $\{\lambda_0, \ldots, \lambda_n\}$ be the set of barycentric coordinates of $S$; i.e., each $\lambda_i \in \mathbb{R}[X]$ is linear and

$$X = \sum_{i=0}^{n} v_i \lambda_i(X), \quad 1 = \sum_{i=0}^{n} \lambda_i(X), \quad \lambda_i(v_j) = \delta_{ij}.$$ 

Given $f \in \mathbb{R}[X]$ of degree $d$, then for any $m \geq d$, there exists a homogeneous polynomial $\tilde{f}_m$ in $n + 1$ variables of degree $m$ such that $\tilde{f}_m(\lambda_0, \ldots, \lambda_n) = f(X)$. We can construct $\tilde{f}_m$ as follows: Suppose $f(X) = \sum_{|\alpha| \leq d} a_{\alpha} X^\alpha$, then

$$f(X) = \sum_{|\alpha| \leq d} a_{\alpha} \left( \sum_{i=0}^{n} v_i \lambda_i(X) \right)^\alpha \left( \sum_{i=0}^{n} \lambda_i(X) \right)^{m-|\alpha}},$$

thus we set

$$\tilde{f}_m(y_0, \ldots, y_n) = \sum_{|\alpha| \leq d} a_{\alpha} \left( \sum_{i=0}^{n} v_i y_i \right)^\alpha \left( \sum_{i=0}^{n} y_i \right)^{m-|\alpha}}. \quad (11)$$

Note that for $d = \text{deg } f$, $\tilde{f}_d$ is the Bernstein-Bézier form of $f$ with respect to $S$.

The following theorem is a generalization of [7, Thm. 6]. Without the concrete bound and with a different proof, it was proven by Michelli and Pinkus [5, 2.6].

**Theorem 3.** Suppose $f \in \mathbb{R}[X]$ of degree $d$ is strictly positive on $S$ and $\tilde{f}_d(y_0, \ldots, y_n)$ is as defined as in (11). Let $\lambda$ be the minimum of $f$ on $S$ and $L = L(\tilde{f}_d)$. Then for

$$N \geq \frac{d(d-1) L}{2} \lambda - d,$$

$f$ has a representation of the form (10) of degree $N$.

**Proof.** Since $f > 0$ on $S$, it follows easily that $\tilde{f}_d > 0$ on $\Delta_{n+1}$. Thus we can apply Pólya’s Theorem to $\tilde{f}_d$ to find $N$ such that $(\sum y_i)^N \tilde{f}_d(Y)$ has positive coefficients. Then we have

$$(\sum y_i)^N \tilde{f}_d(Y) = \sum_{|\beta|=N} b_{\beta} Y^\beta,$$

where $b_{\beta} \geq 0$ for all $\beta$. Substituting $\lambda_i$ for $y_i$ yields $f(X)$ on the left, and a representation of degree $N$ on the right. The bound on $N$ comes from Theorem 1 if we first note that the minimum of $\tilde{f}_d$ on $\Delta_{n+1}$ is the same as the minimum of $f$ on $S$. 

Now we turn to the more general case of the polyhedron $P$ described in the beginning of the section. Given $f$ strictly positive on $P$, we want to use the same technique as for simplices, i.e., find a homogeneous polynomial $g$ which is positive on $\Delta_k$ such that when we “plug in” the $\lambda_i$’s we obtain $f$. In this case, however, finding $g$ is not quite so straightforward.

We fix $P$ as above and $\{\lambda_1, \ldots, \lambda_k\}$ such that $P = \{\lambda_i \geq 0\}$. We first note that by [3], there must exist positive real $c_i$ such that $\sum_i c_i \lambda_i = 1$. The $c_i$’s are found by solving a linear system. We replace each $\lambda_i$ by $c_i \lambda_i$ so that we have

$$\sum_i \lambda_i = 1 \quad (12)$$
Furthermore, there exist constants $b_{i,j} \in \mathbb{R}$ so that, for $j = 1, \ldots, n$,

$$x_j = \sum_{j=1}^{k} b_{i,j} \lambda_i.$$ 

Again, explicitly finding the $b_{i,j}$'s is an easy linear algebra problem. Thus we are almost in the situation for simplices, although the $b_{i,j}$'s need not be positive. Let $B$ be the real $n \times k$ matrix $(b_{i,j})$, then

$$B \cdot (\lambda_1, \ldots, \lambda_k)^T = (x_1, \ldots, x_n)$$

(13)

As in [9], we formalize the notion of “plugging in” the $\lambda_i$'s. Let $\mathbb{R}[Y] := \mathbb{R}[y_1, \ldots, y_k]$ and define $\phi : \mathbb{R}[Y] \to \mathbb{R}[X]$ by $y_i \mapsto \lambda_i$. By (12) and (13), $\phi$ is onto. More explicitly, given a polynomial $f = \sum_{|\alpha| \leq d} a_\alpha X^\alpha$, define homogeneous $\tilde{f} \in \mathbb{R}[Y]$ by

$$\tilde{f} := \sum_{|\alpha| \leq d} a_\alpha (B \cdot Y^T)^\alpha (\sum_{j=1}^{k} y_j)^{|\alpha|}.$$  

(14)

Then $\phi(\tilde{f}) = f$.

Suppose now that $f > 0$ on $P$ and we have a point $\gamma \in \Delta_k$. Then $\tilde{f}(\gamma) = f(B \cdot \gamma)$. Since the point $B \cdot \gamma$ need not be in $P$, we do not necessarily have that $\tilde{f}(\gamma)$ is positive. Thus we cannot apply Polya’s Theorem directly to $\tilde{f}$. However, by a theorem of Schweighofer [9], it turns out that there is a polynomial positive on $\Delta_k$ of the form $\tilde{f} + c(\sum_j r_j^2)$, where $\{r_1, \ldots, r_k\}$ is any basis for the kernel of $\phi$. Note that any $g$ of this form has the property $\phi(g) = f$. The following result is (essentially) [9, Lemma 3.1]:

**Lemma 4.** Suppose $P$ and $\phi$ are as above and $f > 0$ on $P$. Let $\{r_1, \ldots, r_k\}$ be a basis for the kernel of $\phi$, set $r := \sum_j r_j^2$, and define $\tilde{f}$ as in (14). Then for sufficiently large $c$, $\tilde{f} + cr$ is strictly positive on $\Delta_k$. More explicitly, if $\tilde{f}$ is already strictly positive on $\Delta_k$, then we take $c = 0$ and otherwise, this holds for $c > \frac{m_1}{m_2}$, where $m_1$ is the minimum of $\tilde{f}$ on $\Delta_k$ and $m_2$ is the minimum of $r$ on $\Delta_k \cap \{\beta \in \mathbb{R}^k \mid \tilde{f}(\beta) \leq 0\}$.

**Proof.** Let $U$ be the compact set $\Delta_k \cap \{\beta \in \mathbb{R}^k \mid \tilde{f}(\beta) \leq 0\}$ and assume that $U \neq \emptyset$. By [9, §3], $r > 0$ on $U$ and hence, since $U$ is compact, the minimum $m_2$ of $r$ on $U$ exists and is positive. Then on $U, \tilde{f} + cr \geq m_1 + cm_2 > 0$. On $\Delta_k \setminus U, \tilde{f} + cr \geq \tilde{f} > 0$. 

**Theorem 5.** Given $P, \phi, r, f,$ and $\tilde{f}$ as above. Fix $c$ such that $F := \tilde{f} + cr > 0$ on $\Delta_k$. Let $d$ be the degree of $f$ and let $\lambda$ be the minimum of $\tilde{F}$ on $\Delta_k$. For

$$N \geq \frac{d(d-1)}{2} \frac{L(\tilde{F})}{\lambda} - d,$$

$f$ has a representation

$$f = \sum_{|\alpha| = N} b_\alpha \lambda_1^{q_1} \cdots \lambda_k^{q_k},$$

where $b_\alpha \geq 0$ for all $\alpha$.

**Proof.** Since $\phi(\tilde{F}) = f$, this follows from Theorem 1 applied to $\tilde{F}$, exactly as in the proof of Theorem 3. 

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Remark. Note that for a specific $P$ and $f$, we can calculate all elements needed for the bound in the theorem, and then can easily find a representation for $f$. Thus the theorem yields an algorithm for finding a representation for $f$ of the form (10).

Algorithm. Given a compact, convex polyhedron $P \subseteq \mathbb{R}^n$ defined by $\{\lambda_1 \geq 0, \ldots, \lambda_k \geq 0\}$, where $\sum \lambda_i = 1$, and $f > 0$ on $P$. We will describe a procedure for constructing a representation of $f$ of the form (10). We proceed as follows:

1. Using (14), construct homogeneous $\tilde{f} \in \mathbb{R}[Y]$ with the same degree as $f$ such that $\phi(\tilde{f}) = f$.
2. Construct a basis $\{r_1, \ldots, r_\ell\}$ for the kernel of $\phi$. We can use the following well-known procedure for this: Construct a Groebner Basis $G$ for the ideal generated by $\{y_1 - \lambda_1, \ldots, y_k - \lambda_k\}$, using lex order with $x_1 > \cdots > x_n > y_1 > \cdots > y_k$. Then $G \cap \mathbb{R}[y_1, \ldots, y_k]$ is the desired basis.
3. Calculate the minima $m_1$ and $m_2$ needed for the $c$ of Lemma 4, e.g., by using Lagrange multipliers. Set $F := \tilde{f} + cr$ and find the homogeneous $\tilde{F}$.
4. Calculate $L(\tilde{F})$ and the minimum of $\tilde{F}$ on $\Delta_k$ and then find $N$ as in Theorem 5. Use the coefficients of $Y^N\tilde{F}$ to obtain the desired representation.

Example. Let $P$ be the square unit square centered at the origin in $\mathbb{R}^2$, and let $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{1/4 + 1/4 x, 1/4 - 1/4 x, 1/4 + 1/4 y, 1/4 - 1/4 y\}$. With $\phi$ the map defined above, we have that $\{r_1, r_2\} := \{y_1 + y_2 - 1/2, y_3 + y_4 - 1/2\}$ is a Groebner Basis for the kernel of $\phi$. Consider $f := 3/2 - x^2 + y^2 > 0$ on $P$, then

$$\tilde{f} = -\frac{5}{2} y_1^2 + 11 y_1 y_2 - \frac{5}{2} y_2^2 + 3 y_1 y_3 + 3 y_2 y_3 + \frac{11}{2} y_3^2 + 3 y_1 y_4 + 3 y_2 y_4 - 5 y_3 y_4 + \frac{11}{2} y_4^2$$

The minimum of $r := r_1^2 + r_2^2$ on $\{\tilde{f} \leq 0\} \cap \Delta_4$ is 1 and the minimum of $\tilde{f}$ on $\Delta_4$ is $-5/2$. Thus we need $c > 5/2$. We choose $c = 3$, and set $\tilde{F} := \tilde{f} + 3r$. Then

$$\tilde{F} = \frac{7}{2} y_1^2 + 23 y_1 y_2 + \frac{7}{2} y_2^2 - 9 y_1 y_3 - 9 y_2 y_3 + \frac{23}{2} y_3^2 - 9 y_1 y_4 - 9 y_2 y_4 + 7 y_3 y_4 + \frac{23}{2} y_4^2$$

which is positive on $\Delta_4$. The minimum of $\tilde{F}$ on $\Delta_4$ is $3/10$ and $L(\tilde{F}) = 23/2$. Hence the bound in Theorem 1 is 75. This means that that $(y_1 + y_2 + y_3 + y_4)^3 \tilde{F}$ must have positive coefficients. Expanding, and plugging in the $\lambda_i$'s, we could then obtain an explicit representation for $f$. In point of fact, $f$ has an explicit representation of degree 3.

References


