UIUC Department of Mathematics

Geometric Potpourri Seminar

Not all who wander are lost, but that's not the best way to write a paper: an inadvertently hilarious journey in experimental mathematics

Prof. Bruce Reznick
UIUC Department of Mathematics

Abstract: My airplane problem (blank paper, no references) for Fall, 2008 was a simple optimization question. Given $-1 < t_1 < t_2 < \cdots < t_n \le 1$, let M_n be the maximum possible value of $\Pi_{i<j}(t_i - t_j)^2$. What are M_n, the distribution of the t_i's, and the nature of the polynomial $p_n(x) = \Pi(x - t_i)$ for which M_n is the discriminant? One might hope that the t_i's are equally spaced in $[-1, 1]$; for $n \ge 4$, one would be disappointed. For $n = 4$, 5, 6, these nodes are

$$\{\pm 1, \pm 1/\sqrt{5}\}, \qquad \left\{0, \pm 1, \pm\sqrt{\tfrac{3}{7}}\right\}, \qquad \left\{\pm 1, \pm\sqrt{\tfrac{7\pm 2\sqrt{7}}{21}}\right\}.$$

They get worse, but keep the symmetry.

I will describe a trip through recurrences, orthogonal polynomials, and a heartbreaking Google session in a coffeeshop on La Cienega in LA, and how this (and more) was stolen from me by a 19^{th} century son-of-a-Stieltjes. Ultimately, $\{p_n\}$ have a name attached to them, and they are the only polynomials that satisfy the frosh calculus visual lie that polynomials with all real zeros have inflection points precisely at each of their interior zeros. The t_i's are roughly distributed like $\{\cos \frac{k\pi}{n-1} \colon 0 \le k \le n-1\}$, and the M_n's have bizarre factorizations; a typical value is

$$M_{10} = \frac{2^{90} \cdot 7^7}{11^{11} \cdot 13^{13} \cdot 17^{17}}.$$

Some number theorists know these results; I haven't found an algebraic geometer who did. The talk will be as informal as I can make it.

2:00 p.m.
Tuesday, February 24, 2009
243 Altgeld Hall