1 Introductory ideas and examples

A perfectly obvious property of this symbol, that we will use repeatedly, is

\[ [x^n][x^n f(x)] = [x^{n-a}] f(x). \]  

\[ (1.2.7) \]

Another property of this symbol is the convention that if \( \beta \) is any real number, then

\[ [\beta x^n] f(x) = (1/\beta) [x^n] f(x), \]

so, for instance, \( [x^n/n!]e^x = 1 \) for all \( n \geq 0 \).

Before we move on to the next example, here is a summary of the method of generating functions as we have used it so far.

THE METHOD

Given: a recurrence formula that is to be solved by the method of generating functions.

1. Make sure that the set of values of the free variable (say \( n \)) for which the given recurrence relation is true, is clearly delineated.
2. Give a name to the generating function that you will look for, and write out that function in terms of the unknown sequence (e.g., call it \( A(x) \), and define it to be \( \sum_{n \geq 0} a_n x^n \)).
3. Multiply both sides of the recurrence by \( x^n \), and sum over all values of \( n \) for which the recurrence holds.
4. Express both sides of the resulting equation explicitly in terms of your generating function \( A(x) \).
5. Solve the resulting equation for the unknown generating function \( A(x) \).
6. If you want an exact formula for the sequence that is defined by the given recurrence relation, then attempt to get such a formula by expanding \( A(x) \) into a power series by any method you can think of. In particular, if \( A(x) \) is a rational function (quotient of two polynomials), then success will result from expanding in partial fractions and then handling each of the resulting terms separately.

1.3 A three term recurrence

Now let's do the Fibonacci recurrence

\[ F_{n+1} = F_n + F_{n-1}. \quad (n \geq 1; F_0 = 0; F_1 = 1). \]  

\[ (1.3.1) \]

Following 'The Method,' we will solve for the generating function

\[ F(x) = \sum_{n \geq 0} F_n x^n. \]

1.3 A three term recurrence

To do that, multiply \((1.3.1)\) left side

\[ F_2 x + F_2 \]

and on the right side we find \( \{F_1 x + F_2 x^2 + F_3 x^3 + \cdots \} + \)

(Important: Try to do this get the same answer.) It fol

Now we will find some \( \frac{x}{(1 - x - x^2)} \) in partial method is greatly enhanced

\[ 1 - x - x^2 = (1 - x \]

and so

\[ \frac{x}{1 - x - x^2} = \frac{x}{(1 - x \]

\[ = \frac{v}{(1 - x \]

thanks to the magic of the coefficient of \( x^n \) and find

\[ F_n = \frac{1}{\sqrt{5}} (r^n - v^n), \]

as an explicit formula for the

This example offers us a generating functions can tell us the exact answer, but also at

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\[ F_n. \]
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\[ x^{n-\sigma}f(x). \]  

(1.2.7)

\[ 3) \sum_{n=0}^{\infty} a_n x^n, \]  

(1.2.8)

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(1.3.1)

the generating function

\[ n^x. \]

1.9 A three term recurrence

To do that, multiply (1.3.1) by \( x^n \), and sum over \( n \geq 1 \). We find on the left side

\[ F_2x + F_3x^2 + F_4x^3 + \cdots = \frac{F(x) - x}{x}, \]

and on the right side we find

\[ \{F_1x + F_2x^2 + F_3x^3 + \cdots \} + \{F_0x + F_1x^2 + F_2x^3 + \cdots \} = \{F(x)\} + \{xF(x)\}. \]

(Important: Try to do the above yourself, without peeking, and see if you get the same answer.) It follows that \((F - x)/x = F + xF\), and therefore that the unknown generating function is now known, and it is

\[ F(x) = \frac{x}{1 - x - x^2}. \]

Now we will find some formulas for the Fibonacci numbers by expanding \( x/(1 - x - x^2) \) in partial fractions. The success of the partial fraction method is greatly enhanced by having only linear (first degree) factors in the denominator, whereas what we now have is a quadratic factor. So let’s factor it further. We find that

\[ 1 - x - x^2 = (1 - xr_+)(1 - xr_-) \]

\[ (r_+ = (1 \pm \sqrt{5})/2) \]

and so

\[ \frac{x}{1 - x - x^2} = \frac{x}{(1 - xr_+)(1 - xr_-)} \]

\[ = \frac{1}{(r_+ - r_-)} \left( \frac{1}{1 - xr_+} - \frac{1}{1 - xr_-} \right) \]

\[ = \frac{1}{\sqrt{5}} \left\{ \sum_{j=0}^{\infty} r_+^j x^j - \sum_{j=0}^{\infty} r_-^j x^j \right\}. \]

thanks to the magic of the geometric series. It is easy to pick out the coefficient of \( x^n \) and find

\[ F_n = \frac{1}{\sqrt{5}} (r_+^n - r_-^n) \quad (n = 0, 1, 2, \ldots) \]  

(1.3.3)

as an explicit formula for the Fibonacci numbers \( F_n \).

This example offers us a chance to edge a little further into what generating functions can tell us about sequences, in that we can get not only the exact answer, but also an approximate answer, valid when \( n \) is large. Indeed, when \( n \) is large, since \( r_+ > 1 \) and \( |r_-| < 1 \), the second term in (1.3.3) will be minuscule compared to the first, so an extremely good approximation to \( F_n \) will be

\[ F_n \sim \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n. \]  

(1.3.4)