A solution to 2.4–19

To prove $\lim_{x\to 3} \frac{x}{5} = \frac{3}{5}$ by the precise definition of limits, we need to show that for every $\epsilon > 0$, there exists $\delta > 0$ so that

$$\text{if } 0 < |x-3| < \delta, \qquad \text{then } |f(x) - \tfrac{3}{5}| < \epsilon.$$

After some side calculations, we observe that if $\delta = 5\epsilon$, we have that

$$0 < |x-3| < 5\epsilon \implies \tfrac{1}{5}|x-3| < \tfrac{5\epsilon}{\epsilon} \quad \text{or} \quad |\tfrac{x}{5} - \tfrac{3}{5}| < \epsilon.$$

So we can say: "Given $\epsilon > 0$, let $\delta = 5\epsilon$. Then if $0 < |x-3| < \delta$, we have $|f(x) - \frac{3}{5}| < \epsilon$. This completes the proof."