You've proved a theorem, now what? (For Math 296, 8/27/03)

1. The first thing to do is to savor the proof, whether it's true or not. After a decent interval, the next thing to do is to put away the proof you wrote and try to prove it again from scratch. Make sure you understand all the definitions of your terminology: it's very embarrassing to discover that you haven't proved what you thought you proved.

2. Congratulations, you've proved a theorem! What you want to do now is see how you can water it and make it grow.

a. First, try to push the conclusions. Have you made full use of your argument? Have you proved more than you thought?

b. Next, try to pull the hypotheses. Do you need every assumption? Does your argument apply to a more general class of object? Have you proved more than you thought?

c. The aesthetics of most mathematicians favors sharp divisions: "either condition 1 or condition 2", with easily distinguished conditions. For this reason, it is desirable to make as many "if and only if" propositions as possible. Suppose your theorem is of the form $\cup_j A_j \implies \cup_k B_k$. Suggestion a. above is a request for more B_k's; suggestion b. is a request for fewer A_j's; this suggestion is a request for seeing if $\cup_k B_k \implies \cup_j A_j$. Failing that, you can look to see what $A_i \& \cup_k B_k$ implies for the various i's.

d. A theorem isn't very interesting unless it has some applications. Work out the details of your theorem in at least one specific case. If you find that your theorem doesn't answer all the questions in this case, you know the next thing you should work on.

e. A theorem has a natural rhythm. Does your proof remind you in any way of another proof you know, possibly in an entirely different subject? Keep in mind that, in the global overview of human thought, any two branches of mathematics are extremely close to one another. The fabled "unreasonable effectiveness" of mathematics in the physical sciences is easily explained by realizing that nearly the same sorts of minds are necessary to becoming a mathematician or a physical scientist.

f. It always helps to try to explain your work to other people. Not only are they likely to have useful suggestions, but the process of preparing an explanation is extremely valuable in clarifying your own thoughts.

3. Here are some perhaps vaguer guides to research:

a. The *Seinfeld Principle*. Take what you are doing and do the opposite. Switch the foreground and the background. For example, if you are trying to find the roots of a polynomial based on its coefficients (very hard!), instead, try to find the coefficients of a polynomial based on its roots (very useful, and the door to symmetric polynomials!). Note: I know the name is outdated. Any current pop-culture suggestions are welcome.

b. The *Oprah Principle*. Visualize solving your problem and having a solution. Then what? Look for additional properties your solution would have. In this way, you can often make it much easier to solve the original problem.

c. The *Mad Magazine Principle*. Put everything in play. Change, one-by-one, each of your hypotheses and assumptions and see what you can prove. (If you change the rules of implication itself, you are a mathematical logician!)

d. Look for linearity in your problem. If your problem isn't linear, it may be interesting to measure the deviation from linearity.

e. Look for symmetry in your problem. If your problem isn't symmetric, it may be interesting to measure the deviation from symmetry.

f. Ditto for commutativity.

g. Consider turning every number that appears in your problem into a parameter. And remember that 0 and 1 are numbers.

h. If some pair of parameters behaves in a particular way in several examples, see what happens if you enforce this as a hypothesis. (Suggestions g. and h. are paired: you add "degrees of freedom" and then you take them away.)

i. Identify a few texts which have influenced your taste and periodically re-read them. Always remember Pólya's landmark series of books on mathematical problem solving. Insight comes at odd times. Be ready to make notes.

4. Humorist Robert Benchley wrote that there were two kinds of people in the world, those who divide the world into two kinds of people, and those who don't. In the same way, there are many dichotomies in mathematical research. Some are legitimate, some are not.

a1. "Theory-building" versus "Problem-solving" (False dichotomy.) In fact, no collection of theories is very interesting unless it allows you to solve some problems, and no collection of problems is very interesting unless it illustrates some underlying theory.

a2. "Pure" versus "applied" (False dichotomy.) There are too many examples of "pure" mathematical results which are "unreasonably effective" in studying the real world, or more accurately, mathematical models of the real world. On the other hand, there is lots of "applied" mathematics which is so far from reality in its assumptions that it might as well be considered pure.

I think the rest of the dichotomies are legitimate, and reflect genuine differences of mathematical temperament. Be aware of your natural inclinations along these lines, and force yourself to go against them if you are stuck.

b1. "Discrete" versus "continuous". Obvious, but every integer is a real number!

b2. "Static" versus "dynamic". Do you see mathematical objects acting on each other as you watch, or do you see them already having had their influence?

b3. "Equality" versus "isomorphism". What exactly do you mean when you say things are "equal"?

b4. "Abstract" versus "concrete". Almost a false dichotomy, because every concrete mathematical object is really abstract if you think about it foundationally, and if you look at any abstract object long enough, it will seem to be concrete.

b5. "Intuitive" versus "formal". Like b4., this is almost a false dichotomy. Perhaps a better version would be "visualizable" versus "axiomatic".

b6. "Finite" versus "infinite". This underestimates the complications involved. There are many "finite" questions using numbers so large as to be "practically" infinite, although the differences (e.g. compactness) are crucial. There are many gradations of "infinite".

b7. "Coordinates" versus "intrinsic". This is a specialization of "analytic" versus "geometric". Do you give names to all the pieces of your problem and follow them along, or do you go with a gestalt? Do you think in pictures or equations?

This final set of dichotomies is really psychological.

b8. "Inner-directed" versus "outer-directed". (Here, I mean whether you are answering your own question, or questions that someone told you to answer. This may be the most profound difference between coursework and research, apart from the difficulty of the problems.)

b9. "Hedgehog" versus "fox". (Based on the classical Greek poet Archilochus who wrote: "The fox knows many things, but the hedgehog knows one big thing'.) In the best of all worlds, you would know many big things.

Comments and additional suggestions are always welcome.