Math 347 **Homework 8** **Due Wed., Nov. 5, 2003**

(There aren't a lot of questions on sections 21 and 22 because their is thematically different from the rest of the course.)

1. – 20.11 a,c (ungraded).

2. – 21.3 (ungraded).

3. – 23.1 a,c,e,g (ungraded).

4. – 20.11b.

5. – 21.8.

6. – 23.1 b,d,f,h.

7. – ($\mathcal{E}$). Determine the radius of convergence and the interval of convergence of

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^n(x-1)^n}{n} = 2(x-1) - \frac{4(x-1)^2}{2} + \frac{8(x-1)^3}{3} - \frac{16(x-1)^4}{4} + \ldots.$$

8. – ($\mathcal{E}$). Three unrelated and possibly simple-minded true-false problems – proof or simple counterexample required:

a. If f is a function defined on $(0,1)$ and for all $x \in (0,1)$ $|f(x)| < 1/x$, then f is unbounded on $(0,1)$.

b. The function $f(x) = \sin\frac{1}{x}$ is not uniformly continuous on $(0,1)$.

c. If f_n is a sequence of continuous functions defined on $[0,1]$ and $f_n \to 0$ uniformly on $[0,1]$, then $\sum f_n(x)$ is convergent for all $x \in [0,1]$.

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9. – 23.4, but with $a_n = \left(\frac{5+3(-1)^n}{7}\right)^n$, so that $a_{2n} = (\frac{8}{7})^{2n}$ and $a_{2n+1} = (\frac{2}{7})^{2n+1}$.

10. – #9 (continued). **Assume** the following theorem (proved on the back): if $\sum_{n=0}^{\infty} a_n$ is absolutely convergent, then $\sum a_{2k}$ and $\sum a_{2k+1}$ are both also absolutely convergent and

$$(*) \qquad \sum_{n=0}^{\infty} a_n = \sum_{k=0}^{\infty} a_{2k} + \sum_{k=0}^{\infty} a_{2k+1}.$$

Using (*), find a closed form for the power series in #9 on its interval of absolute convergence.

Proof of (*) – just for your information. Let

$$t_N = \sum_{n=0}^{N} |a_n|.$$

Since $\sum a_n$ is absolutely convergent, we know that $t_N \uparrow M := \sum |a_n|$.

Observe that

$$\sum_{n=0}^{N} |a_{2n}| = |a_0| + |a_2| + \cdots + |a_{2N}| \leq \sum_{n=0}^{2N} |a_n| = t_{2N} \leq M,$$

and

$$\sum_{n=0}^{N} |a_{2n+1}| = |a_1| + |a_3| + \cdots + |a_{2N+1}| \leq \sum_{n=0}^{2N+1} |a_n| = t_{2N+1} \leq M.$$

Since the partial sums of $\sum |a_{2n}|$ and $\sum |a_{2n+1}|$ are non-decreasing (by definition) and bounded above (by M), the series are convergent, and hence $\sum a_{2n}$ and $\sum a_{2n+1}$ are both absolutely convergent, and hence convergent.

Suppose $\sum a_{2n} = L$ and $\sum a_{2n+1} = L'$, for real L and L'. Given $\epsilon > 0$. there exist N_1 and N_2 so that

$$N > N_1 \implies \left| \sum_{n=0}^{N} a_{2n} - L \right| < \frac{\epsilon}{2}, \qquad N > N_2 \implies \left| \sum_{n=0}^{N} a_{2n+1} - L' \right| < \frac{\epsilon}{2},$$

Write $N_0 := \max\{2N_1, 2N_2+1\}$. By breaking up the partial sums of the original sequence into the terms with even index and odd index. By the usual inequalities, we find that

$$N > N_0 \implies \left| \sum_{n=0}^{N} a_n - (L + L') \right| < \epsilon.$$

QED