1. In order (using De Moivre's Theorem and other basics),

$$
\begin{aligned}
(1+i)^2 &= 1+2i+i^2 = 1+2i-1 = 2i;\\
(1+i)^{11} &= \left(\sqrt{2}e^{i\pi/4}\right)^{11} = \sqrt{2}^{11}e^{11i\pi/4} = 32\sqrt{2}\left(-\tfrac{\sqrt{2}}{2}+i\tfrac{\sqrt{2}}{2}\right) = -32+32i;\\
\frac{3+4i}{1-2i} &= \frac{(3+4i)(1+2i)}{(1-2i)(1+2i)} = \frac{3-8+(4+6)i}{1+4} = -1+2i;\\
z^3 &= (x+iy)^3 = x^3+3x^2yi+3xy^2(-1)+y^3(-i) = (x^3-3xy^2)+i(3x^2y-y^3);\\
\bar{z}z &= (x-iy)(x+iy) = x^2+y^2;\\
\frac{\bar{z}}{z} &= \frac{x-iy}{x+iy} = \frac{(x-iy)^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2} - i\frac{2xy}{x^2+y^2};\\
\frac{z-i}{1-\bar{z}i} &= \frac{x+i(y-1)}{1-i(x-iy)} = \frac{x+i(y-1)}{(1-y)-ix} = \frac{(x+i(y-1))((1-y)+ix)}{(1-y)^2+x^2}\\
&= \frac{2x(1-y)}{(1-y)^2+x^2} + i\frac{x^2-(y-1)^2}{(1-y)^2+x^2}.
\end{aligned}
$$

2. (corrected!) We have, conveniently enough, that

$$\frac{1+5i}{2-3i} = \frac{(1+5i)(2+3i)}{(2-3i)(2+3i)} = \frac{2-15+(3+10)i}{4+9} = -1+i = \sqrt{2}e^{3i\pi/4}.$$

Thus the desired answer is

$$2^{-168}2^{348/2}e^{3*348i\pi/4} = 2^{174-168}e^{261i\pi} = -64.$$

Mathematica gives the answer to the original problem as, roughly, $-42.5856-47.7752i$. Expanded as rational numbers, the real and imaginary parts would not fit into one line.

3. We have $-4 = 4e^{\pi i}$, so $z^4 = -4$, hence

$$
\begin{aligned}
z = 4^{1/4}e^{\pi i/4},\quad & 4^{1/4}e^{\pi i/4+\pi i/2},\quad 4^{1/4}e^{\pi i/4+\pi i},\quad 4^{1/4}e^{\pi i/4+3\pi i/2}\\
&= 1+i,\quad -1+i,\quad -1-i,\quad 1-i.
\end{aligned}
$$

Similarly, $8i = 8e^{\pi i/2}$, so that the three cube roots are

$$2e^{\pi i/6},\quad 2e^{\pi i/6+2\pi i/3},\quad 2e^{\pi i/6+4\pi i/3} = \sqrt{3}+i,\quad -\sqrt{3}+i,\quad -2i.$$

Finally, if $z^{-2} = 2i$, then $z^2 = \frac{1}{2i} = -\frac{i}{2} = \frac{1}{2}e^{3\pi i/2}$, so that the values of z are

$$\frac{1}{\sqrt{2}}e^{3\pi i/4},\quad \frac{1}{\sqrt{2}}e^{3\pi i/4+\pi i} = -\frac{1}{2}+\frac{i}{2},\quad \frac{1}{2}-\frac{i}{2}.$$

4. In order, these can be described as: a vertical line, the right half-plane, the unit circle, the outside of the unit circle, a horizontal line, the half-plane below this line and an annulus. Pictures are ok as well, to "count" as a description.

5. See pictures; note that $-1 = e^{\pi i}$, $1+i = \sqrt{2}e^{i\pi/4}$, $\frac{1}{1-i} = \frac{1+i}{2}$.

6. If $f(z) = \frac{1+z}{1-z}$, then

$$f(z+1) = \frac{1+z+1}{1-z-1} = -\frac{2+z}{z}; \qquad f(\tfrac{1}{z}) = \frac{1+\frac{1}{z}}{1-\frac{1}{z}} = \frac{z+1}{z-1} \qquad (= -f(z));$$

$$f(f(z)) = \frac{1+\frac{1+z}{1-z}}{1-\frac{1+z}{1-z}} = \frac{1-z+1+z}{1-z-(1+z)} = \frac{2}{-2z} = -\frac{1}{z}.$$

7. (revised) This is the strip bounded by the lines $x=-1$ and $x=1$. The first transformation doubles the region and shifts it up by 1 (not that we'd notice), into the region bounded by $x=-2$ and $x=2$. The second transformation can't be done so easily in words. If $z = 1+it$, then $w = (1+i)(1+it)+1 = 2-t+i(1+t)$, and the pair $(2-t, 1+t)$ parameterizes the line $x+y=3$. Similarly, if $z=-1+it$, then $w = (1+i)(-1+it)+1 = -t+i(-1-t)$, and the pair $(-t,-1-t)$ parameterizes the line $x+y=-1$: the region is the strip between these two lines.

8. See picture. Note that if $z = re^{i\alpha}$ and $w = 1/z$, then $w = (1/r)e^{-i\alpha}$.

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9. Notice that $P_1(x) = x$ and since $(z+z^{-1})^2 = z^2+2+z^{-2}$, we have $P_2(x) = x^2-2$. Suppose by induction that $z^n + z^{-n} = P_n(z+z^{-1})$. Let $f_n(z) = z+z^{-1}$, and observe that

$$f_n(z)f_1(z) = \left(z^n + \frac{1}{z^n}\right)\left(z+\frac{1}{z}\right) = \left(z^{n+1} + \frac{1}{z^{n+1}}\right) + \left(z^{n-1} + \frac{1}{z^{n-1}}\right) = f_{n+1}(z) - f_{n-1}(z).$$

That is, $(z+\frac{1}{z})P_n(z+\frac{1}{z}) - P_{n-1}(z+\frac{1}{z}) = z^{n+1} + \frac{1}{z^{n+1}}$. Thus, we may take $P_{n+1}(x) = xP_n(x) - P_{n-1}(x)$, and so by induction, P_{n+1} is a polynomial of degree $n+1$.

This is closely related to a famous family of polynomials called the Chebyshev polynomials, and we'll see later that P_n has the interesting property that $P_n(2\cos(\theta)) = 2\cos(n\theta)$.

An alternative proof is to solve for z in the equation $z + z^{-1} = w$; this becomes a quadratic in z, $z^2 - wz + 1 = 0$,and there are two solutions; $z = \frac{w \pm \sqrt{w^2-4}}{2}$. However, the solutions are reciprocals, so in any event,

$$z^n + \frac{1}{z^n} = \left(\frac{w + \sqrt{w^2 - 4}}{2}\right)^n + \left(\frac{w - \sqrt{w^2 - 4}}{2}\right)^n .$$

If you now expand the right-hand side by the binomial theorem, and note that $(a+b)^n + (a-b)^n = 2\sum_k \binom{n}{2k} a^{n-2k} b^{2k}$, because alternate terms cancel, you will get an explicit formula:

$$z^n + \frac{1}{z^n} = 2 \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \left(z + \frac{1}{z}\right)^{n-2k} \left((z + \frac{1}{z})^2 - 4\right)^k .$$

10. You can substitute in directly, or note that $z^2 = x^2 - y^2 + i2xy$, so that $\bar{z}^2 = x^2 - y^2 - i2xy$, and so $x^2 - y^2 = \frac{1}{2}(z^2 + \bar{z}^2)$ and $2xy = \frac{1}{2i}(z^2 - \bar{z}^2)$, and similarly, $x^3 - 3xy^2 = \frac{1}{2}(z^3 + \bar{z}^3)$.

11. There are two ways to draw the triangle. In the first, $(z_3 - z_2) = \omega(z_2 - z_1)$ and in the second, $(z_3 - z_2) = \omega^2(z_2 - z_1)$. Thus, recalling that $\omega + \omega^2 = -1$, either

$$z_3 - z_2 - \omega(z_2 - z_1) = \omega z_1 - (1 + \omega) z_2 + z_3 = \omega z_1 + \omega^2 z_2 + z_3 = 0$$

or

$$z_3 - z_2 - \omega^2(z_2 - z_1) = \omega^2 z_1 - (1 + \omega^2) z_2 + z_3 = \omega^2 z_1 + \omega z_2 + z_3 = 0$$

Either one happens if and only if the product of the two polynomials is 0; that is

$$0 = (\omega z_1 + \omega^2 z_2 + z_3)(\omega^2 z_1 + \omega z_2 + z_3) = \omega^3 z_1^2 + \omega^3 z_2^2 + z_3^2$$
$$+(\omega^2 + \omega^4) z_1 z_2 + (\omega + \omega^2)(z_1 z_3 + z_2 z_3) = z_1^2 + z_2^2 + z_3^2 - (z_1 z_2 + z_1 z_3 + z_2 z_3).$$

12. Writing $a_n + ib_n = (3 + 4i)^n$, we have

$$a_{n+1} + ib_{n+1} = (a_n + ib_n)(3 + 4i) = (3a_n - 4b_n) + i(4a_n + 3b_n).$$

If you know modular arithmetic, it's easy to see that $a_n \equiv 3 \mod 5$ and $b_n \equiv 4 \mod 5$ imply the same for a_{n+1} and b_{n+1}. If you don't know modular arithmetic, follow the hint and find that

$$5c_{n+1} + 3 = 3(5c_n + 3) - 4(5d_n + 4) = 5(3c_n - 4d_n - 2) + 3 \implies c_{n+1} = 3c_n - 4d_n - 2$$
$$5d_{n+1} + 4 = 4(5c_n + 3) + 3(5d_n + 4) = 5(4c_n + 3d_n + 4) + 4 \implies d_{n+1} = 4c_n + 3d_n + 4$$

and induction implies that c_n and d_n are integers for $n \geq 1$.

If $\frac{\arctan 4/3}{\pi} = \frac{m}{n}$, then $z = 3 + 4i = 5e^{mi\pi/n}$, so that $z^n = (3+4i)^n = 5^n e^{m\pi i} = \pm 5^n$ would be real, and so that $b_n = 0$, but this is impossible since $b_n \equiv 4 \mod 5$.