1. For $z = Re^{it}$, $Log(z) = \log R + it$, so $|Log(z)| \le R + |t| \le R + \pi$. (A better upper bound is $\sqrt{R^2+\pi^2}$.) Combining this with the usual estimates based on the size of the numerator and denominator, and the length of the contour, we have

$$\left| \int_{C_{2,R}} \frac{z^2 Log(z)}{1+z^8}\, dz \right| \le \pi R \cdot \frac{R^2}{R^8 - 1}(R+\pi) := \phi(R) < \frac{c}{R^4} \to 0.$$

I'll accept the usual variations on this (with the better upper bound above, or with $(1-\epsilon)R^8$ in the denominator if desired.)

2. Let $g(z) = \frac{10z}{2+z}$. Then $|g(z)| \ge \frac{10}{3}$ for $|z| = 1$, and so $|g(z)| > |f(z)|$ (comfortably!) on $|z| = 1$. Rouché's Theorem says that g and $g - f$ have the same number of zeros inside $|z| = 1$. But $g(z_0) = 0 \iff z_0 = 0$. Thus $g - f$ also has only one zero inside the unit circle. But $(g-f)(z_0) = 0 \iff \frac{10z_0}{2+z_0} = f(z_0)$. Multiplicity is impossible. (Note that the singularity of g is at -2, which is outside the circle.)

3. The mindless calculation converges only for $|\frac{1}{z}| < 1$; that is, for $|z| > 1$. Accordingly, the given series is not the Laurent series for $f(z)$ at $z = 0$, and so its structure is irrelevant for the analysis of f at $z = 0$. Two remarks: the given series actually shows that f has a removable singularity at ∞, and the correct series at $z = 0$ is:

$$\frac{z}{z-1} = -\frac{z}{1-z} = -z - z^2 - z^3 - \cdots .$$

4. This problem uses the Pac-Man contour. Let $f(z) = \frac{z^{1/2}\log z}{(1+z)^2}$. Both functions in the numerator need branch cuts, which usually go on the negative real axis. However, there is a pole there (at $z = -1$), which makes me want to make a branch cut on the positive real axis. Thus, for $z = re^{it}$, $0 \le t < 2\pi$, we have $\log z = \log r + it$ and $z^{1/2} = r^{1/2}e^{it/2}$. So I'm going to take the following contour, with parameters ϵ, R and θ, where ϵ and θ are small and R is large. We first (C_1) go on the line $z = re^{i\theta}$ as r goes from ϵ to R. Then (C_2) we go on the circle $z = Re^{it}$ as t goes from θ to $2\pi - \theta$. Then (C_3), we go on the line $z = re^{i(2\pi-\theta)}$ as r goes from R to ϵ, and then we go on the circle $z = \epsilon e^{it}$ as t goes from $2\pi - \theta$ to θ. This contour circles $z = 0$ to its left, so that the only singularity is the double pole at $z = -1$. We have

$$\int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz + \int_{C_3} f(z)\, dz + \int_{C_4} f(z)\, dz = 2\pi i \cdot Res(f(z), -1).$$

Let $g(z) = z^{1/2}\log z$, so that $g'(z) = \frac{1}{2}z^{-1/2}\log z + z^{1/2}\frac{1}{z} = \frac{\log z + 2}{2z^{1/2}}$; the residue is $g'(-1) = \frac{\pi i + 2}{2i} = \pi/2 - i$. Now it's time to look at the contours; as $\epsilon, \theta \to 0, R \to \infty$,

$$\int_{C_1} \frac{z^{1/2}\log z}{(1+z)^2}\, dz = \int_\epsilon^R \frac{x^{1/2}e^{i\theta/2}(\log x + i\theta)}{(1+e^{i\theta}x)^2} e^{i\theta}\, dx \to \int_0^\infty \frac{x^{1/2}\log x}{(1+x)^2}\, dx;$$

$$\left| \int_{C_2} \frac{z^{1/2}\log z}{(1+z)^2} \right| < \frac{R^{1/2}(\log R + \pi)}{(R-1)^2} \cdot 2\pi R \to 0;$$

$$\int_{C_3} \frac{z^{1/2}\log z}{(1+z)^2}\, dz = \int_R^\epsilon \frac{x^{1/2}e^{i(2\pi-\theta)/2}(\log x + i(2\pi-\theta))}{(1+e^{i(2\pi-\theta)}x)^2} e^{i(2\pi-\theta)}\, dx \to \int_0^\infty \frac{x^{1/2}(\log x + 2\pi i)}{(1+x)^2}\, dx;$$

$$\left| \int_{C_4} \frac{z^{1/2}\log z}{(1+z)^2}\, dz \right| \le \frac{\epsilon^{1/2}(|\log \epsilon| + 2\pi)}{(1-\epsilon)^2} \cdot 2\pi\epsilon \to 0.$$

Putting this together, we see that

$$\int_0^\infty \left(\frac{x^{1/2}\log x}{(1+x)^2} + \frac{x^{1/2}(\log x + 2\pi i)}{(1+x)^2} \right) dx = 2\pi i(\tfrac{\pi}{2} - i) = 2\pi + \pi^2 i.$$

Comparing the real and the imaginary parts, we get the desired values from

$$\int_0^\infty \frac{2x^{1/2}\log x}{(1+x)^2}\, dx = 2\pi; \quad \int_0^\infty \frac{x^{1/2} 2\pi}{(1+x)^2}\, dx = \pi^2.$$

5. The point of this problem is to see that you can make substitutions for an ugly integral. We have $z = e^{\pi i}x = -x$, where x runs from 10 to 1, so

$$\int_C z^{3/4}(log(z))^2\, dz = \int_{x=10}^1 (-x)^{3/4}(log(-x))^2\, d(-x) = \int_{x=1}^{10} x^{3/4}e^{3\pi i/4}(\log x + i\pi)^2\, dx$$

$$\int_{x=1}^{10} x^{3/4}\left(\frac{-1+i}{\sqrt{2}}\right)\left(\log^2 x - \pi^2 + (2\pi \log x)i\right)\, dx.$$

By working out the real and imaginary parts of the integrand, we get

$$g(x) = \frac{1}{\sqrt{2}}x^{3/4}\left(-\log^2 x + \pi^2 - 2\pi\log x\right), \qquad h(x) = \frac{1}{\sqrt{2}}x^{3/4}\left(\log^2 x - \pi^2 - 2\pi\log x\right).$$

6. Following the hint, we know that for R sufficiently large (i.e., $|R| > |z_k|$), we have

$$\int_{|z|=R} f(z)\, dz = 2\pi i \sum_{k=1}^{n} Res(f(z), z_k)$$

On the other hand, for R sufficiently large, $|z| = R$ implies that $|f(z)| < \frac{2}{R^n}$ [actually $\frac{\tau}{R^n}$ will do for any $\tau > 1$] hence the integral is bounded in absolute value by $\frac{4\pi}{R^{n-1}}$. Since this is true for all sufficiently large R, the sum of the residues must be zero.

7. Since $T(-1) = \infty$ and $T(1) = 0$, we have $T(z) = \lambda\left(\frac{z-1}{z+1}\right)$ for some λ, hence $1 + i = T(i) = \lambda\left(\frac{i-1}{i+1}\right) = i\lambda$, so $\lambda = 1 - i$ and

$$T(z) = (1-i)\left(\frac{z-1}{z+1}\right).$$

Since 1, -1 and i are on the unit circle, the image of the unit circle is determined by their images: 0, ∞ and $1+i$. This is the line $y = x$. To determine the image of the real axis, we need the images of three points on the real line. We already know two of them: $T(-1) = \infty$ and $T(1) = 0$; since $T(0) = -1 + i$, we see that the image is the line $y = -x$. It is also correct to observe that $T(\infty) = 1 - i$, since ∞ is contained on every line. This gives the same conclusion.

These can also be found directly: if z is on the unit circle, then $z = e^{it}$, so that

$$T(z) = (1-i)\left(\frac{e^{it}-1}{e^{it}+1}\right) = (1-i)\left(\frac{e^{it/2}-e^{-it/2}}{e^{it/2}+e^{-it/2}}\right) = (1-i)\frac{2i\sin(t/2)}{2\cos(t/2)} = (1+i)\tan(t/2),$$

and we see a parameterization of the line $y = x$. Similarly, if $z = x$ is real, then $T(z) = \left(\frac{x-1}{x+1}\right)(1-i)$ and we see a parameterization of the line $y = -x$.

8. This is the classic implication of Rouché's Theorem: Suppose $f(z) = z^n$ and $g(z) = a_{n-1}z^{n-1} + \cdots + a_0$ and let $R = \sum_k |a_k| + 1$. We have already seen that all the roots of $f(z) + g(z)$ must lie inside $|z| < R$ via the estimate that, for $|z| \geq R$ (≥ 1),

$$\left|\frac{f(z)}{g(z)}\right| = |z|^{-n}|g(z)| \leq \sum_{k=0}^{n-1} |a_k||z^{k-n}| \leq \sum_{k=0}^{n-1} |a_k||z^{-1}| \leq \frac{1}{R}\sum_{k=0}^{n-1} |a_k| < 1.$$

But this also implies that $|f(z)| > |g(z)|$ on $|z| = R$, so that $f + g$ and f have the same number of zeros inside $|z| = R$, counting multiplicity. Since f has n zeros, this shows that $f + g$ has n zeros, counting multiplicity, which is the Fundamental Theorem of Algebra.

9. If $w = f(z) = z^2 + 2z$, then $f'(z) = 2z + 2$, so f is conformal for every $z \neq -1$. When $z = i$, $f'(i) = 2i + 2 = \sqrt{8}e^{i\pi/4}$. Thus the scale factor is $\sqrt{8}$, and the angle of rotation is $\pi/4$. Note: we have $f(z-1) = z^2 - 1$, so that $f(-1 + ae^{it}) = a^2e^{2it} - 1$. This shows explicitly how the map is not conformal: angles are doubled.

10. Since $|e^z| = e^x$ for $z = x + iy$, we have $e^{-r} \leq |e^z| \leq e^r$ when $|z| = r$. Thus, on $z = .7$, $|e^z| \geq e^{-.7} = .4965... > .343 = .7^3$. Thus, $|e^z| > |z^3|$, so $f(z)$ and e^z have the same number of zeros inside $|z| = .7$; that is, zero. On the other hand, for $z = 2$, $|e^z| \leq e^2 = 7.389... < 8 = 2^3$, so $f(z)$ and z^3 have the same number of zeros inside. $z| = 2$, counting multiplicity, namely 3.

I didn't make it clear, but you can show that these zeros are simple, so that the multiplicity is 1. Here's how to see that: if $f(z_0) = f'(z_0) = 0$, then

$$0 = e^{z_0} + z_0^3 = e^{z_0} + 3z_0^2 \implies z_0^3 = 3z_0^2 \implies z_0 \in \{0, 3\},$$

and $f(0) = 1, f(3) = e^3 + 27$ shows that these aren't zeros. Mathematica gives three solutions in the complex plane: $z \approx -.77288, .18463 \pm 1.04733i$.

11. Let

$$T(z) = \frac{z + 2i}{z + i}.$$

Then, $T(1) = \frac{1+2i}{1+i} = \frac{3}{2} + \frac{i}{2}$, $T(i) = \frac{i+2i}{i+i} = \frac{3}{2}$, $T(0) = \frac{2i}{i} = 2$, $T(1+i) = \frac{1+3i}{1+2i} = \frac{7}{5} + \frac{i}{5}$, and

$$\mathcal{X}(1, i, 0, 1+i) = \frac{(1-i)(-(1+i))}{-i(-i)} = \frac{-2}{-1} = 2$$

$$\mathcal{X}(1.5 + .5i, 1.5, 2, 1.4 + .2i) = \frac{(.5i)(.6 - .2i)}{(.1 + .3i)(.5)} = \frac{.1 + .3i}{(.1 + .3i)(.5)} = 2.$$

Yup, the cross-ratio is preserved.

12. Here's the gruesome detail: Since $f(0) = 0$ and $f'(0) \neq 0$, it follows that f is not identically zero and $z = 0$ is a zero of order one. Hence, $f(z) = zh(z)$ with $h(0) \neq 0$ and since h is analytic, it's continuous, and we can say that $h(z) \neq 0$ for $|z| < \rho$ for some positive ρ, which we can take $\leq R$, and so $f(z) \neq 0$. Let $0 < \delta = \min\{|f(z)| : |z| = \rho\}$. Suppose α is such that $|\alpha| < \delta$ and $g(z)$ is the constant function $-\alpha$. Then $|f(z)| > |g(z)|$ on $|z| = \rho$, so f and $f + g$ have the same number of zeros inside $z = \rho$. We've seen that f has exactly one zero, at $z = 0$, hence $(f + g)(z) = f(z) - \alpha$ has exactly one zero. (Multiplicity is ruled out by the count.) The import of this problem will be clearer later this week.