1. Lots of ways to do this, and I am writing more than I expect you to. Let $T(z) = \frac{z}{z+1}$. To determine the image of the real axis, we need three or four easy points. I picked:

$$T(-1) = \infty, \qquad T(0) = 0, \qquad T(1) = \frac{1}{2}, \qquad T(\infty) = 1$$

This means that $T : \mathbf{R} \to \mathbf{R}$. (Also true by class discussion.) By continuity, T maps $[0, \infty] \to [0, 1]$ We already know that 0 and ∞ are also on the imaginary axis, so we need one (but will compute two) additional points:

$$T(i) = \frac{i}{1+i} = \frac{i}{1+i} \cdot \frac{1-i}{1-i} = \frac{1+i}{2}; \qquad T(-i) = \frac{-i}{1-i} = \frac{-i}{1-i} \cdot \frac{1+i}{1+i} = \frac{1-i}{2}$$

So $T : \{0, \infty, i\} \to \{0, 1, \frac{1+i}{2}\}$. It's easy to see that the circle C which passes through these three points has center $\frac{1}{2}$ and radius $\frac{1}{2}$. The image of the first quadrant is then one of the four regions bounded by C and the real axis. By looking at the location of the points $0, 1, \infty, i, 0$ as they run on the boundary of the first quadrant, we see that this image has to be the upper half-disk bounded by C and the real axis. As a check, let's take a point inside the first quadrant:

$$T(1+i) = \frac{1+i}{2+i} = \frac{1+i}{2+i} \cdot \frac{2-i}{2-i} = \frac{3+i}{5}$$

As a check $|T(i+i) - \frac{1}{2}| = \sqrt{.1^2 + .2^2} = \sqrt{.05} < \frac{1}{2}$. See picture at end.

2. We need to show that $f(z) = (1-i)\left(\frac{z-i}{z-1}\right)$ maps the unit circle to the real axis, and one point inside the unit circle to the upper half plane. We see immediately that $f(i) = 0$ and $f(1) = \infty$. There are two other obvious choices for points on the unit circle, and

$$f(-1) = (1-i) \cdot \frac{-1-i}{-1-1} = \frac{-2}{-2} = 1; \qquad f(-i) = (1-i) \cdot \frac{-i-i}{-i-1} = (-2i)i = 2.$$

Thus f maps the unit circle to the real axis. Furthermore,

$$f(0) = (1-i) \cdot \frac{0-i}{0-1} = (1-i)i = 1+i$$

is in the upper half plane, so we're done.

3. This is very much like example 3.1 on p. 287. Recall that the function $\phi(x, y) = \frac{1}{\pi} Arg z$ has the pleasant property of being harmonic, and taking the value 0 on the positive real axis and 1 on the negative real axis. It is undefined of course at $z = 0, \infty$. If we compose this function with $f(z)$ from 2., we see that

$$T(x, y) = (\phi \circ f)(x + iy) = \frac{1}{\pi} Arg \left((1-i) \left(\frac{z-i}{z-1} \right) \right)$$

has the following properties: if $|z| < 1$, then since $f(z)$ is in the upper half plane, $T(x, y)$ will be a harmonic function taking a value between 0 and 1. Furthermore, since f takes $1, i, -1, -i$ in that order to $\infty, 0, 1, 2$, we see that the quarter-arc from 1 to i must be mapped to the negative real axis, and the three-quarter arc from i to 1 must be mapped to the positive real axis, and so if we look at $T(\cos\theta, \sin\theta)$ for $\theta \in (0, \frac{\pi}{2})$, we are looking at $\phi(x, 0)$ for negative x, which is 1, and if $\theta \in (\frac{\pi}{2}, 2\pi)$, we are looking at $\phi(x, 0)$ for positive x, which is 0. To make an explicit function, observe that

$$(1-i)\left(\frac{z-i}{z-1}\right) = (1-i)\cdot\frac{x+i(y-1)}{(x-1)+iy} = \frac{(x^2+y^2-2x-2y+1)+i(1-x^2-y^2)}{(x-1)^2+y^2},$$

so that

$$T(x,y) = \frac{1}{\pi}\arctan\left(\frac{1-x^2-y^2}{x^2+y^2-2x-2y+1}\right),$$

with arctan defined on $(0, \pi)$. As a check, you can see that on the circle, $1 - x^2 - y^2 = 0$, so the imaginary part of $(1-i)(\frac{z-1}{z+1})$ is zero, and the real part is $2 - 2x - 2y$ which changes sign where the line $x + y = 1$ crosses the unit circle, at $(1, 0)$ and $(0, 1)$.

4. Let $T(z) = \frac{i}{z+1}$. The line $x = c$ contains the points $z = c$ and $z = \infty$ (since ∞ is on every line). Thus, the image of the line contains the points $\frac{i}{c+1}$ and 0. In general,

$$T(c+iy) = \frac{i}{c+1+iy} = \frac{i}{c+1+iy}\cdot\frac{c+1-iy}{c+1-iy} = \frac{y+i(c+1)}{(c+1)^2+y^2}.$$

Thus $u = \frac{y}{(c+1)^2+y^2}$ and $v = \frac{c+1}{(c+1)^2+y^2}$ and

$$u^2+v^2 = \frac{y^2+(c+1)^2}{((c+1)^2+y^2)^2} = \frac{1}{(c+1)^2+y^2} = \frac{v}{c+1} \implies u^2+v^2-\frac{v}{c+1} = 0.$$

Thus, the image is a circle with center $(0, \frac{1}{2(c+1)})$ and radius $\frac{1}{2(c+1)}$.

5. The region given is bounded by the three line segments from 0 to 1 to ∞ to 0 and

$$T(i) = \frac{i}{i+1} = \frac{1+i}{2}, \quad T(0) = i, \quad T(1) = \frac{i}{2}, \quad T(1+i) = \frac{i}{2+i} = \frac{1+2i}{5}, \quad T(\infty) = 0.$$

Using 4., the image of the full imaginary axis is the circle with center $(0, \frac{1}{2})$ with radius $\frac{1}{2}$, the image of the line $y = 1$ is the circle with center $(0, \frac{1}{4})$ with radius $\frac{1}{4}$. We have $T(0) = i, T(1) = \frac{i}{2}$ and $T(\infty) = 0$, hence the image of the real axis is the imaginary axis. See picture at the end.

6. §5.6-3a. If $f(z) = \frac{z}{(1-z)^2}$, then

$$f(z_1) - f(z_2) = \frac{z_1}{(1-z_1)^2} - \frac{z_2}{(1-z_2)^2} = \frac{z_1(1-z_2)^2 - z_2(1-z_1)^2}{(1-z_1)(1-z_2)^2} = \frac{(z_1-z_2)(1-z_1z_2)}{(1-z_1)(1-z_2)^2}.$$

Thus, if $f(z_1) = f(z_2)$, then either $z_1 = z_2$ or $z_1 z_2 = 1$. Thus, if $|z_1|, |z_2| < 1$, then $z_1 = z_2$. This is a famous example, because $f(z) = \sum_{n=0}^{\infty} nz^n$. The "Bieberbach Conjecture", proved by Louis de Branges of Purdue in the 80's, assserted that, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ was simple in the unit circle, then $|a_n| \leq n$ for all n. It was once a major open question.

§5.6-3b. The given region is three-quarters of the annulus $1 < |z| < 2$. The image of the region under $w = z^2$ is the full annulus $1 < |w| < 4$, which is not only not simply connected, but the function isn't simple: for example, $(\sqrt{2}i)^2 = -2 = (-\sqrt{2}i)^2$.

7. The main problem here is that one cannot say that the integral over the semi-circle goes to zero, because, if $f(z) = \frac{e^{\pi z}}{1+z^2}$ and $z = Re^{i\theta}$, then

$$|e^{\pi z}| = e^{\pi R \cos\theta},$$

which will be quite large for θ near zero. For any $R > 1$, it *is* correct to say that

$$\int_{-R}^{R} \frac{e^{\pi x}}{1+x^2}\, dx + \int_{C_{2,R}} \frac{e^{\pi z}}{1+z^2}\, dz = 2\pi i \cdot \frac{e^{\pi i}}{2i} = -\pi,$$

and the fact that the first integral goes to plus infinity as $R \to \infty$ can in fact be used to show that the second integral goes to minus infinity.

8. As noted, let

$$F(a,b) = \int_{-\infty}^{\infty} \frac{x\sin(ax)}{x^2+b^2}\, dx.$$

We evaluate $F(a,b)$ by integrating $f(z) = \frac{ze^{iaz}}{z^2+b^2}$ over the protractor contour, and since the degree of the polynomial in the numerator is 1, and that of the polynomial in the denominator is 2, we've already shown that the integral over the semicircle goes to 0 as $R \to \infty$. Therefore, since f has a simple pole at $z = bi$,

$$\int_{-\infty}^{\infty} \frac{x(\cos(ax) + i\sin(ax))}{x^2+b^2}\, dx = 2\pi i \cdot \frac{bie^{-ab}}{2bi} = \pi e^{-ab} i.$$

Taking the imaginary part above, we get $F(a,b) = \pi e^{-ab}$, which is symmetric in a and b. (The real part gives the integral of an odd function, which is 0, as it should be.)

9. We have $\frac{\partial F}{\partial b}(a,b) = -a\pi e^{-ab}$. If we take the partial derivative inside the integral in 8., we would want to calculate

$$\int_{-\infty}^{\infty} \frac{-2bx\sin(ax)}{(x^2+b^2)^2}\, dx,$$

which sugggests that we integrate $g(z) = \frac{-2bze^{iaz}}{(z^2+b^2)^2}$ over the same contour. Since the denominator has an even higher degree than before, the integral over the semicircle goes to zero. We have

$$g(z) = \frac{h(z)}{(z-bi)^2}; \qquad h(z) = \frac{-2bze^{iaz}}{(z+bi)^2} \implies h'(z) = \frac{-2be^{iaz} - 2iabze^{iaz}}{(z+bi)^2} + \frac{4bze^{iaz}}{(z+bi)^3}$$

By the same argument as above, (the denominator now has an even higher degree, but the pole is now order 2),

$$\int_{-\infty}^{\infty} (-2b) \frac{x(\cos(ax) + i\sin(ax))}{(x^2+b^2)^2}\, dx = 2\pi i h'(bi)$$
$$= 2\pi i \left(\frac{e^{-ab}(-2b+2ab^2)}{-4b^2} + \frac{4b^2 i e^{-ab}}{-8b^3 i} \right) = 2\pi i e^{-ab} \left(\frac{1}{2b} - \frac{a}{2} - \frac{1}{2b} \right) = -a\pi i e^{-ab}.$$

Taking the imaginary part, we obtain the desired integral.

10. The integrand is not integrable, because $\frac{x^2\cos(ax)}{x^2+1}$ behaves like $\cos(ax)$ for large x, which doesn't go to zero. It is not hard to show that the integrals over $[(n-\frac{1}{2})\pi, (n+\frac{1}{2})\pi]$ do not go to zero, so by definition the improper integral is undefined. Furthermore, we do not know that the integral over the semicircle goes to zero, since for large values of R the natural integrand is $\frac{\partial f}{\partial a}(z) = \frac{iz^2 e^{iaz}}{z^2+b^2}$, which is bounded, but the integral doesn't go to zero. It's important to note for later work that there is no way to tell from $F(a,b)$ alone whether it's legal to differentiate under the integral sign.

11. Following the hint, suppose f is analytic in a convex region, and suppose z_1 and z_2 are in the domain. We are given that $|f'(z) - 1| < 1$ on the domain. We want to show that $f(z_1) \neq f(z_2)$. Write the line segment from z_1 to z_2 as $\zeta = z_1 + t(z_2 - z_1)$. We have

$$f(z_2) - f(z_1) = \int_C f'(\zeta)\, d\zeta = \int_{t=0}^{1} f'\big(z_1 + t(z_2 - z_1)\big)(z_2 - z_1)\, dt$$
$$= \big(z_2 - z_1\big) \left[\left(\int_{t=0}^{1} f'\big(z_1 + t(z_2 - z_1)\big) - 1\, dt \right) + \left(\int_{t=0}^{1} 1\, dt \right) \right].$$

If $f(z_2) = f(z_1)$, then we can factor out the constant $z_2 - z_1$ above and conclude that

$$\left(\int_{t=0}^{1} \big(f'\big(z_1 + t(z_2 - z_1)\big) - 1\big)\, dt \right) = -1.$$

Taking the absolute value gives a contradiction. What's really going on is this: the given condition also implies that $Re(f'(z)) > 0$ for all z in the domain, so if you look at

$$\frac{f(z_1) - f(z_2)}{z_1 - z_2} = \int_{t=0}^{1} f'(z_1 + t(z_2 - z_1))\, dt,$$

the right side is an integral with positive real part, and so it can't vanish.

12. Thank you for your work this semester.