Math 348 **Homework 2 Solutions** **Monday, Feb. 7, 2000**

1. So by now, you know to look at the image of the points on lines of the form $z = a+it$, where $-1 < a < 1$. If $w = 2z^2$, then $w = u + iv = 2(a+it)^2 = 2(a^2 - t^2) + i(4at)$, so $u = 2(a^2 - t^2), v = 4at$.

If $a = 0$, then $v = 0$ and $u = -2t^2$. This says that if z is on the positive imaginary axis, then w is on the negative real axis. Fair enough! If $a \neq 0$, then $t = \frac{v}{4a}$, hence $u = 2a^2 - \frac{1}{8a^2}u^2$. This gives a parabola with vertex $(2a^2, 0)$ and hairpinning out towards the negative u-axis. The image of the strip is contained to the inside of the parabolas corresponding either to $a = 1$ or $a = -1$. This is a case where both lines on the boundary map to the same set; it's a little tricky.

Suppose $w = 1/z$ and $z = a+it$, then $w = u+iv = \frac{1}{a+it}$, so $u = \frac{a}{a^2+t^2}, v = \frac{-t}{a^2+t^2}$, and as in class, we have

$$u^2 + v^2 = \frac{a^2+t^2}{(a^2+t^2)^2} = \frac{1}{a^2+t^2} = \frac{u}{a} \qquad \text{if } a \neq 0.$$

If $a = 0$, then $(u, v) = (0, -1/t)$, which parameterizes the entire v-axis. If $a \neq 0$, then $u^2 + v^2 = u/a$, which gives the circle:

$$\left(u - \frac{1}{2a}\right)^2 + v^2 = \left(\frac{1}{2a}\right)^2$$

with center at $(\frac{1}{2a}, 0)$ and radius $\frac{1}{2a}$; that is to say, the circle with this center which passes through the origin. If you draw some of these and throw in the case with $a = 0$, you will see that the image is the entire complex plane, except for the two circles with center $(\pm\frac{1}{2}, 0)$ and radius $\frac{1}{2}$. This might not seem intuitive, except that everything in $|Re(z)| > 1$ has modulus at least 1, and so $|w| \leq 1$. If I find more succinct solutions, I will pass them along.

2. When $z \neq z_0$, these expressions reduce to

$$1, \qquad z^3 + z_0 z^2 + z_0^2 z + z_0^3, \qquad -\frac{1}{zz_0}, \qquad -\frac{z+z_0}{z^2 z_0^2}.$$

and it's easy to see that as $z \to z_0$, the limits are $1, 4z_0^3, -\frac{1}{z_0^2}, -\frac{2}{z_0^3}$, as they ought to be.

3. The definition of bounded on p.26 says that S is bounded if and only if there exists M so that $z \in S$ implies that $|z| \leq M$. If S is bounded, and z_0 is given, then by the triangle inequality, if $z \in S$, then

$$|z - z_0| \leq |z| + |z_0| \leq M + |z_0|,$$

so for any T so that $T > M + |z_0$, we have $|z - z_0| < T$ for every $z \in S$. Conversely, if $|z - z_0| < T$ for every $z \in S$, then we have

$$|z| = |(z - z_0) + z_0| < |z - z_0| + |z_0| < T + |z_0|,$$

so if we take $M = T + |z_0|$, then we see that the definition on p.26 is satisfied.

4. I'd do the first one by the chain rule, and the second one by the quotient rule:

$$\Big((2z+3)^5\Big)' = 5(2z+3)^4 * 2; \qquad \left(\frac{z-i}{z+i}\right)' = \frac{(z+i)*1 - (z-i)*1}{(z+i)^2} = \frac{2i}{(z+i)^2}.$$

5. If $u = x^2 - y^2$ and $v = -2xy$, then

$$u_x = 2x \neq v_y = -2x; \qquad u_y = -2y \neq -v_x = -(-2y).$$

Both fail. (This function is $\bar{z}^2$.)

If $u = x^3 - 3y^2 + 2x$ and $v = 3x^2y - y^3 + 2y$, then

$$u_x = 3x^2 + 2 \neq v_y = 3y^2 - 3x^2 + 2; \qquad u_y = -6y \neq -v_x = -(6xy).$$

Almost. I'm willing to bet that this was a misprint. If $-3y^2$ is replaced by $-3y^2x$, then the equations are satisfied (and $f(z) = z^3 + 2z$.)

6. Written in order (and you can also do these using formulas in the book):

$$\begin{aligned}
\cosh \pi i &= \frac{e^{\pi i} + e^{-\pi i}}{2} = -1 + 0 * i; e^i = \cos(1) + i\sin(1); \\
\sin i\pi &= \frac{e^{i(i\pi)} - e^{-i(i\pi)}}{2i} = \frac{e^{-\pi} - e^{\pi}}{2i} = i\sinh(\pi); \\
\sinh(1+i) &= \frac{e^{1+i} - e^{-1-i}}{2} = \frac{1}{2}\left((e\cos 1 + ie\sin 1 - e^{-1}\cos(-1) - ie^{-1}\sin(-1)\right) \\
&= \cos 1 \frac{e - e^{-1}}{2} + i\sin 1 \frac{e + e^{-1}}{2} = \cos 1 \sinh 1 + i \sin 1 \cosh 1.
\end{aligned}$$

7. Throughout, n denotes an arbitrary integer.

$$\begin{aligned}
e^z = -4 = 4e^{i\pi} \iff z = \log 4 + i\pi + i(2n\pi); \\
\cos z = -4 \iff \frac{e^{iz} + e^{-iz}}{2} = -4 \iff \left(e^{iz}\right)^2 + 8e^{iz} + 1 = 0 \iff e^{iz} = -4 \pm \sqrt{15} \\
\iff e^{iz} = \left(4 \pm \sqrt{15}\right) e^{i\pi} \iff iz = \log(4 \pm \sqrt{15}) + i\pi + 2n\pi i \iff z = \pi + 2n\pi - i\log(4 \pm \sqrt{15}); \\
\sin z = -4 \iff \frac{e^{iz} - e^{-iz}}{2i} = -4 \iff \left(e^{iz}\right)^2 + 8ie^{iz} - 1 = 0 \iff e^{iz} = \frac{-8i \pm \sqrt{-64+4}}{2} = (-4 \pm \sqrt{15})i \\
= (4 \mp \sqrt{15})e^{3\pi i/2} \iff iz = \log(4 \pm \sqrt{15}) + i3\pi/2 + 2n\pi i \iff z = 3\pi/2 + 2n\pi - i\log(4 \pm \sqrt{15}).
\end{aligned}$$

Remarks: notice that, as in the real case, $\sin(z + \pi/2) = \cos z$, and also $\log(4 - \sqrt{15}) = -\log(4 + \sqrt{15})$, which simplifies the expression of the answer.

8. Similar to #7, and it's ok if you used the logarithm notation:

$$e^z = i = e^{\pi i/2} \iff z = \frac{\pi i}{2} + 2n\pi i; e^z = 1 + i\sqrt{3} = 2e^{\pi i/3} \iff z = \log 2 + \frac{\pi i}{3} + 2n\pi i.$$

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9. If $P = (0,0,1)$, $Q = (X, Y, Z)$ and $R = (x, y, 0)$ are collinear (misspelled in the book!), then $\overline{PQ}$ and $\overline{PR}$ are parallel, and so there exists $t \neq 0$ so that

$$\begin{aligned}
(X, Y, Z - 1) = t(x, y, -1) \implies X = tx,\ Y = ty,\ Z = 1 - t \implies \\
1 = (tx)^2 + (ty)^2 + (1-t)^2 \implies 1 = t^2(x^2 + y^2 + 1) - 2t + 1.
\end{aligned}$$

We cancel 1's and divide by t to get $t = \frac{2}{x^2+y^2-1} = \frac{2}{|z|^2+1}$. Direct substitution gives the desired formulas for X, Y, Z.

10. We have

$$\begin{aligned}
a(z)^2 + b(z)^2 \\
= \left(\cos^2 z * a^2 + 2\cos z \sin z * ab + \sin^2 z * b^2\right) + \left(\sin^2 z * a^2 - 2\cos z \sin z * ab + \cos^2 z * b^2\right) \\
= (\sin^2 z + \cos^2 z)(a^2 + b^2) = a^2 + b^2.
\end{aligned}$$

To solve the equation $b(z_0) = 0$, we must have

$$\sin(z_0)a = \cos(z_0)b \iff \frac{e^{iz_0} - e^{-iz_0}}{2i}a = \frac{e^{iz_0} + e^{-iz_0}}{2}b \iff e^{iz_0}(a - ib) = e^{-iz_0}(a + ib).$$

If $a^2 + b^2 = (a + ib)(a - ib) \neq 0$, then we have

$$e^{2iz_0} = \frac{a + ib}{a - ib},$$

which gives a finite ($a - ib \neq 0$) and non-zero ($a + ib \neq 0$) value for e^{2iz_0}. Thus, we get two values for e^{iz_0}:

$$e^{2iz_0} = \frac{(a + ib)^2}{a^2 + b^2} \implies e^{iz_0} = \pm\left(\frac{a + ib}{(a^2 + b^2)^{1/2}}\right).$$

Since a and b may not be real, it is not necessarily true that $(a^2 + b^2)^{1/2} = |a + ib|$! This problem actually came up in my research.

11. Follow the suggestions here. Suppose $|f(z_1) - f(z_2)| = |z_1 - z_2|$ for all z_1, z_2 and suppose $g(z) = \alpha f(z) + \beta$, where $|\alpha| = 1$. Then $|g(z_1) - g(z_2)| = |\alpha f(z_1) + \beta - (\alpha f(z_2) + \beta)| = |\alpha||f(z_1) - f(z_2)| = |z_1 - z_2|$, so g is also an isometry. Since $|f(1) - f(0)| = 1$ because f is an isometry, we may let $\alpha = \frac{1}{f(1) - f(0)}$ and $\beta = \frac{-f(0)}{f(1) - f(0)}$ above and conclude that

$$g(z) = \frac{f(z) - f(0)}{f(1) - f(0)}$$

is an isometry. These weren't picked at random; they make sure that $g(0) = 0$ and $g(1) = 1$. Let $z = x + iy$, where $z \neq 0, 1$ and $w = f(z) = u + iv$. Considering the pairs $(z_1, z_2) = (0, z)$ and $(1, z)$ successively, and squaring the moduli in the definition of isometry, we get:

$$x^2 + y^2 = u^2 + v^2; \qquad (x - 1)^2 + y^2 = (u - 1)^2 + v^2$$

By subtracting these equations, we find that $2x = 2u$, hence $u = x$, and it then follows that $v^2 = y^2$, so $v = \pm y$.

The argument now becomes a little subtle, as we have to show that the choice of sign above is consistent for *all* z. Suppose first that for $z = i$, we have $w = i$; that is, $g(i) = i$. Following the hint, with $(z_1, z_2) = (i, z)$, we get

$$x^2 + (y - 1)^2 = u^2 + (v - 1)^2.$$

Since we already know that $u = x$ and $v^2 = y^2$, this equation implies that $v = y$. Thus, $g(z) = z$ for all z. (Actually for $z \neq 0, 1$, but we've already taken care of these cases earlier!). Similarly, if $g(i) = -i$, then we'll find that $v = -y$, so that $g(z) = \bar{z}$. Working backwards, we see that, up to arbitrary α with $|\alpha| = 1$ and β, any isometry of the plane has the form $\alpha z + \beta$ or $\alpha\bar{z} + \beta$. Speaking in purely geometric terms, this says that all isometries of the plane consist of some combination of rotations, reflections and translations.

12. I'll throw in problem 6. We have

$$\cos z + i\sin z = \frac{e^{iz} + e^{-iz}}{2} + i\frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2} + \frac{e^{iz} - e^{-iz}}{2} = e^{iz}.$$

Thus, the addition formula for the cosine amounts to the addition formula $e^{iz_1 + iz_2} = e^{iz_1}e^{iz_2}$:

$$\cos(z_1 + z_2) + i\sin(z_1 + z_2) = e^{iz_1 + iz_2} = e^{iz_1}e^{iz_2} = (\cos z_1 + i\sin z_1)(\cos z_2 + i\sin z_2).$$

Replace z_1 and z_2 by $-z_1$ and $-z_2$ and note that $\cos(-z) = \cos(z)$, but $\sin(-z) = -\sin(z)$, to get

$$\cos(z_1 + z_2) - i\sin(z_1 + z_2) = e^{-iz_1 - iz_2} = e^{-iz_1}e^{-iz_2} = (\cos z_1 - i\sin z_1)(\cos z_2 - i\sin z_2).$$

It is critical to observe that this equation is **not** found by taking the complex conjugate of the previous one, because the various sines and cosines are complex numbers in general and not real! If you add (and then subtract) these equations, and cancel the appropriate terms, you get

$$2\cos(z_1 + z_2) = 2\cos(z_1)\cos(z_2) - 2\sin(z_1)\sin(z_2),$$
$$2i\sin(z_1 + z_2) = 2i\cos(z_1)\sin(z_2) + 2i\sin(z_1)\cos(z_2).$$