1a. Since $u(x,y) = x^2 - y$ and $v(x,y) = x - y^2$ are each continuous functions for all (x,y), it follows that f is continuous for all complex z.
1b. We've seen that f is differentiable at z_0 if the Cauchy-Riemann equations are satisfied there. It is easy to check that

$$u_x(x_0,y_0) = 2x_0, \quad u_y(x_0,y_0) = -1, \quad v_x(x_0,y_0) = 1, \quad v_y(x_0,y_0) = -2y_0.$$

Thus the C-R equations hold at (x_0,y_0) whenever $2x_0 = -2y_0, -1 = -1$; that is, when $x_0 + y_0 = 0$. That is, f is differentiable on the line $x_0 + y_0 = 0$.
1c. Since a line contains no open sets, f is analytic nowhere.

2. Well, I got a little further in class than I thought, so we've already talked about this in class: $z = z_0 + Re^{i\theta}$, z_0 complex and $R > 0$ constant, and $0 \le \theta < 2\pi$ represents the circle of radius R, centered at z_0 and traversed once counterclockwise.

3. It's OK to write $\tan z$ in terms of $\sin z$ and $\cos z$, and differentiate by the quotient rule. For variety, I went back to the definition, which works provided $\cos z \neq 0$; that is, if and only if $z \neq (n + \frac{1}{2})\pi$:

$$\begin{aligned}
\tan z &= \frac{\sin z}{\cos z} = \frac{\frac{e^{iz}-e^{-iz}}{2i}}{\frac{e^{iz}+e^{-iz}}{2}} = -i \cdot \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = i \cdot \frac{1 - e^{2iz}}{1 + e^{2iz}} \\
\implies (\tan z)' &= i \cdot \frac{(1 + e^{2iz})(-2ie^{2iz}) - (1 - e^{2iz})(2ie^{2iz})}{(1 + e^{2iz})^2} \\
&= i \cdot \frac{-4ie^{2iz}}{(1 + e^{2iz})^2} = \frac{4e^{2iz}}{(1 + e^{2iz})^2} = \frac{4}{(e^{iz} + e^{-iz})^2} = \frac{1}{\cos^2 z}.
\end{aligned}$$

Since (comparing with p.56), we have $\sinh z = -i \sin iz$ and $\cosh z = \cos iz$, it follows that

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{-i \sin iz}{\cos iz} = -i \tan(iz).$$

Thus, by the chain rule,

$$(\tanh z)' = -i \tan'(iz) \cdot i = \frac{1}{\cos^2(iz)} = \frac{1}{\cosh^2(z)}.$$

4. This one is strictly by the definition. First observe that $2 + 2i = 2^{3/2}e^{i\pi/4}$. Therefore

$$\begin{aligned}
(2 + 2i)^i = e^{i \log(2+2i)} &= e^{i((3/2)\log 2 + i\pi/4 + 2\pi i n)} = e^{-\pi/4 - 2\pi n + i(3/2)\log 2} \\
&= e^{-\pi/4 - 2\pi n}\big(\cos(3/2)\log 2 + i \sin(3/2)\log 2\big).
\end{aligned}$$

5. Since $\sin^2 z + \cos^2 z = 1$, the common value is not zero, and this is equivalent to $\tan z = 1$. Using the expansion of $\tan z$ from problem 3 to save time, we have

$$\begin{aligned}
1 = i\frac{1 - e^{2iz}}{1 + e^{2iz}} &\iff 1 + e^{2iz} = i(1 - e^{2iz}) \iff e^{2iz}(1 + i) = i - 1 \\
&\iff e^{2iz} = i \iff 2iz = i(\tfrac{\pi}{2} + 2n\pi) \iff z = \tfrac{\pi}{4} + n\pi.
\end{aligned}$$

In other words, $\sin z = \cos z$ implies that z is real and $\tan z = 1$.

6. The first three are multiple valued, with n representing an arbitrary integer.

$$\log(-i) = \log(e^{-\pi i/2}) = -\frac{i\pi}{2} + 2\pi n i;$$
$$\log(1+i) = \log(\sqrt{2}e^{\pi i/4}) = \frac{\log 2}{2} + \frac{i\pi}{4} + 2\pi n i;$$
$$(-i)^{-i} = e^{(-i)\log(-i)} = e^{(-i)(-\frac{i\pi}{2}+2\pi n i} e^{-\pi/2+2\pi n)};$$
$$i^2 = e^{2\log i} = e^{2(\pi i/2 + n\pi i)} = e^{\pi i + 4n\pi i} = -1.$$

7. The region can be described as those $z = re^{i\theta}$ with $r \geq 2$ and $0 \leq \theta \leq \pi/4$. (Or $r > 2$, depending on how you read the problem.) Accordingly, $w = iz^2 = e^{i\pi/2}r^2e^{2i\theta} = r^2e^{i(2\theta+\pi/2)}$ is a complex number with modulus ≥ 4 and argument between $\pi/2 + 0 = \pi/2$ and $\pi/2 + 2 * \pi/4 = \pi$. And the principal value of the logarithm is $\log r + i\theta$, since θ as given above is the Argument of z. Sketches are below:

8. So, if $z = x + iy = re^{i\theta}$, then $\log z = \log r + i(\theta + 2n\pi)$. Thus,

$$z^i = e^{i\log z} = e^{i\log r - \theta - 2n\pi},$$

so to answer the question,

$$Re(z^i) = e^{-\theta-2n\pi}\cos(\log r); \qquad Im(z^i) = e^{-\theta-2n\pi}\sin(\log r).$$

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9. Because $\overline{z + iw} = \overline{z} - i\overline{w}$,

$$|z+iw|^2 = (z+iw)(\overline{z+iw}) = (z+iw)(\overline{z} - i\overline{w}) = |z|^2 + |w|^2 + i(w\overline{z} - \overline{w}z).$$

Thus, the desired condition is equivalent to $w\overline{z} = \overline{w}z$. There are several ways to proceed from here. One is to note that $\overline{w\overline{z}} = \overline{w}z$, hence this condition is that $w\overline{z} = t$ is real. Multiplying through by z, we get $w|z|^2 = tz \implies w = \frac{t}{|z|^2}z$, so that w is a real multiple of z. Another way is to divide by wz and obtain $\frac{\overline{w}}{w} = \frac{\overline{z}}{z}$. If $w = re^{i\alpha}$ and $z = Re^{i\beta}$, then this implies that $e^{-2i\alpha} = e^{-2i\beta}$. This then implies that $e^{i(\beta-\alpha)} = \pm 1$ and $z = \pm\frac{R}{r}w$.

10. Write $0 \neq z = re^{iArg(z)}$. Then $Log(z) = \log r + iArg(z)$, and $-\pi \leq Arg(z) \leq \pi$. Since $|iz| = r$ as well,

$$f(z) = \log r + iArg(iz) - \log r - iArg(z) = i\big(Arg(iz) - Arg(z)\big).$$

A glance of the definition of the argument shows that $f(z) = i\pi/2$, provided $-\pi < Arg(z) \leq \pi/2$ and $f(z) = -3i\pi/2$, provided $\pi/2 < Arg(z) \leq \pi$. Any choice of points from the appropriate quadrants is acceptable.

11. This is a subtle and not-very-well-worded problem in the text, some of it is more easily explained at the blackboard than on paper. I will grade accordingly. There are two relevant copies of the complex plane at work here: the w-plane and the z-plane.

As a start, we have $w = e^{2it} - 1$, $0 \leq t < 2\pi$. This is a circle of radius 1, centered at -1, and traversed twice in the usual counterclockwise direction. Then $z = w^2$, so we have to figure out what the image of this circle looks like in the z-plane: it turns out to be a cardioid. [I'll show in class how you can rough this out without mathematica.]

The wrinkle is that $z = w^2$ means that w is a square root of z, and this gives a "double covering" of the w-plane on which every $w \neq 0$ appears on two sheets — we take the arguments so they cycle after 4π, not 2π, and this means that the circle appears once on one sheet $0 \leq t < \pi$ and once on the other sheet $\pi \leq t < 2\pi$. If you can draw this, you're better than me!

12. Write $w = re^{i\theta}$, where $\theta = Arg(z)$ is the principal value of the argument, and let $\epsilon = e^{2\pi i/n} \neq 1$, what is called a "primitive" n-th root of unity, then as we've seen, the n-th roots of w have the form

$$r^{1/n}e^{i\theta/n}\epsilon^k, \qquad k \in \{0, 1, \ldots, n-1\}.$$

The sum of the roots is therefore, by summing a geometric series

$$r^{1/n}e^{i\theta/n}(1 + \epsilon + \epsilon^2 + \cdots = \epsilon^{n-1} = r^{1/n}e^{i\theta/n}\frac{\epsilon^n - 1}{\epsilon - 1} = 0.$$

Another way is to know the more general fact that

$$p(z) = z^n - a_{n-1}z^{n-1} + \cdots = a_0 = \prod_{k=1}^{n}(z - z_k) \implies a_{n-1} = \sum_{k=1}^{n} z_k,$$

which is clear if you expand out the product of the factors. The n-th roots of w are simply the roots of $z^n - w$, and if $n \geq 2$, then the coefficient of z^{n-1} above is zero.