1. If $u(x,y) = x^2 - y^2 + y$, then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$, so u is harmonic.
If $u(x,y) = x^3 - y^3$, then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6y$, so u is not harmonic.
If $u(x,y) = 3x^2y - y^3 + xy$, then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y - 6y = 0$, so u is harmonic.
If $u(x,y) = x^4 - 6x^2y^2 + y^4 + x^3y - xy^3$, then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (12x^2 - 12y^2 + 6xy) + (-12x^2 + 12y^2 - 6xy) = 0$, so u is harmonic.

2. If $u(x,y) = x^3 + ax^2y + bxy^2 + y^3$, then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (6x + 2ay) + (2bx + 6y)$, so u is harmonic precisely when $a = b = -3$ and $u(x,y) = x^3 - 3x^2y - 3xy^2 + y^3$.

The harmonic conjugate v will have to satisfy the Cauchy-Riemann equations, so that:

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 3x^2 + 6xy - 3y^2; \qquad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 6xy - 3y^2.$$

The first of these equations implies that $v(x,y) = x^3 + 3x^2y - 3xy^2 + \phi(y)$, and the second then implies that $3x^2 - 6xy + \phi'(y) = 3x^2 - 6xy - 3y^2$, so that $\phi(y) = -y^3 + c$ and $v(x,y) = x^3 + 3x^2y - 3xy^2 - y^3 + c$. The condition $v(0,0) = 1$ implies that $c = 1$.

Finally,

$$\begin{aligned} f(z) &= u(x,y) + iv(x,y) = x^3 - 3x^2y - 3xy^2 + y^3 + i(x^3 + 3x^2y - 3xy^2 - y^3) + 1 \\ &= (1+i)(x+iy)^3 + 1 = (1+i)(x^3 + 3ix^2y - 3xy^2 - iy^3) + 1 = (1+i)z^3 + 1. \end{aligned}$$

3. If $n \neq -1$, then $\left(\frac{z^{n+1}}{n+1}\right)' = z^n$, hence for any closed contour C, Theorem 1.1 implies that $\int_c z^n\, dz = 0$. (Note that if $n \leq -2$, then this function is analytic except at 0, so the Theorem applies to a domain D which avoids $z = 0$. The reason this doesn't apply when $n = -1$ is that there is **no** branch of the logarithm which is defined in a "punctured" neighborhood of 0.) It's old hat by now, but the specific integral requested is

$$\int_C \frac{1}{z}\, dz = \int_0^{2\pi} \frac{1}{e^{it}}\, ie^{it}\, dt = 2\pi i.$$

4. The first arc is a line segment from 1 to $1+i$; the second is a circle traversed clockwise from 1 to -1 in the lower half plane; the third is a closed circle of radius 3, centered at the origin and traversed counterclockwise; the fourth takes the unit circle twice around counterclockwise; the fifth describes a parabola from 1 to $2+i$ with the equation $x = 1 + y^2$.

5. If $x = 1 + it$, $0 \leq t \leq 1$, then for any function f,

$$\int_C f(z)\, dz = \int_{t=0}^{1} f(1+it)\, idt.$$

Thus,

$$\int_C z\, dz = i\int_0^1 (1+it)\, dt = i(t + i\tfrac{t^2}{2})\Big|_{t=0}^{1} = i(1 + \tfrac{i}{2}) = -\tfrac{1}{2} + i.$$

It is also OK to observe that this is merely $\frac{1}{2}z^2|_{z=1}^{1+i}$. Similarly,

$$\int_C z\, dz = i\int_0^1 (1+it)^2\, dt = \int_0^1 (i - 2t - it^2)\, dt = it - t^2 - \tfrac{it^3}{3}\Big|_0^1 = -1 + \tfrac{2i}{3}.$$

Again, this can be done by taking the antiderivative, and computing $\frac{1}{3}z^3|_{z=1}^{1+i} = \frac{(1+i)^3 - 1^3}{3}$.

If $x = 1 + it + t^2$, $0 \le t \le 1$, then for any function f,

$$\int_C f(z)\ dz = \int_{t=0}^{1} f(1 + t^2 + it)\ (2t + i)\ dt.$$

It's much easier to evaluate the integrals using Theorem 1.1, since this is an arc from 1 to $2 + i$, and both z and z^2 are analytic. We obtain then

$$\int_C z\ dz = \tfrac{1}{2} z^2 \Big|_{z=1}^{2+i} = \tfrac{1}{2}\big((2+i)^2 - 1\big) = 1 + 2i;$$
$$\int_C z^2\ dz = \tfrac{1}{3} z^3 \Big|_{z=1}^{2+i} = \tfrac{1}{3}\big((2+i)^3 - 1\big) = \tfrac{1+11i}{3}.$$

6. The recommended contour is $z = e^{it}$, $-\pi/2 \le t \le \pi/2$, so that $dz = ie^{it}\ dt$ as usual. Thus

$$\int_C z^2\ dz = \int_{-\pi/2}^{\pi/2} e^{2it} ie^{it}\ dt = \tfrac{i}{3i} e^{3it} \Big|_{-\pi/2}^{\pi/2} = \tfrac{1}{3}\big(i^3 - (-i)^3\big) = -\tfrac{2i}{3};$$
$$\int_C \overline{z}\ dz = \int_{-\pi/2}^{\pi/2} e^{-it} ie^{it}\ dt = \int_{-\pi/2}^{\pi/2} i\ dt = \pi i.$$

The first integral can also be done using $\frac{z^3}{3}$ as an antiderivative, since z^2 is analytic. No such luck with $\overline{z}$.

7. The line $x = a$ is parameterized by $x + iy = a + it$, so that if $w = z^2$, then $u = a^2 - t^2$, $v = 2at$, and if $a \ne 0$, then as we've seen, $u = a^2 - \left(\frac{v}{2a}\right)^2$ is a parabola. (If $a = 0$, then the image is the negative real axis.) Notice that the images are the same for $x = a$ and $x = -a$, so the image of R is found by taking $0 \le a \le 3$, and is sketched below; the image of the lines for $0 < a \le 2$ is covered twice.

Similarly, if $w = 1/z$, then $u = \frac{a}{a^2+t^2}$, $v = \frac{-t}{a^2+t^2}$, hence if $a \ne 0$, then $u^2 + v^2 = \frac{1}{a} v$ gives the circle with center $(-\frac{1}{2a}, 0)$ and with radius $\frac{1}{2a}$. (If $a = 0$, we get the entire imaginary axis except the origin.)

8. The integrals are pretty routine, and parameterized by $z = e^{it}$, with $0 \le t \le 2\pi$:

$$\int_C \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{it}\ dt}{e^{it}} = 2\pi i;$$
$$\int_C \frac{dz}{|z|} = \int_0^{2\pi} \frac{ie^{it}\ dt}{1} = e^{it}\Big|_0^{2\pi} = 1 - 1 = 0;$$
$$\int_C \frac{dz}{z^2} = \int_0^{2\pi} \frac{ie^{it}\ dt}{e^{2it}} = \int_0^{2\pi} ie^{-it}\ dt = -e^{-it}\Big|_0^{2\pi} = -1 + 1 = 0.$$

The second shows that it is possible for a non-analytic function to integrate to 0 over a closed contour, and the third could also be done as the integral of $(-\frac{1}{z})'$ over a closed contour.

9. We have

$$\int_{C_m} f(z)\, dz = \int_{t=0}^{1} f(t+it^m)\,(1+imt^{m-1})\, dt.$$

Each C_m runs from 0 to $1+i$. So the first integral can be evaluated by an antiderivative, and

$$\int_{C_m} z\, dz = \tfrac{1}{2}z^2\Big|_{z=0}^{1+i} = \frac{(1+i)^2}{2} = i.$$

We have to work harder with $\overline{z}$, which equals $t - it^m$ on the contour:

$$\int_{C_m} \overline{z}\, dz = \int_{t=0}^{1} (t-it^m)(1+imt^{m-1})\, dt = \int_{t=0}^{1} (t+mt^{2m-1}) + i(mt^m - t^m)\, dt$$
$$= \left(\tfrac{1}{2}(t^2+t^{2m}) + i\tfrac{m-1}{m+1}t^{m+1}\right)\Big|_{t=0}^{1} = 1 + i\tfrac{m-1}{m+1}.$$

Unlike the first case, we see that this integral definitely depends on the contour.

10. If $z = \cos t + i\sin t$ is on the unit circle, then

$$|4+3z|^2 = (4+3\cos t)^2 + (3\sin t)^2 = 25 + 24\cos t.$$

Thus, we can certainly say that $|4+3z|^2 \geq 25 - 24 = 1$, hence $|\frac{1}{4+3z}| \leq \frac{1}{1}$ on C, and since C has length 2π, the first estimate is immediate. For the second estimate, we write $C = C_1 \cup C_2$, where C_1 denotes the portion of the unit circle in the right-hand plane and C_2 is the portion in the left hand plane. We still have $|\frac{1}{4+3z}| \leq \frac{1}{1}$ on C_2, but for C_1, $\cos t \geq 0$, hence $|4+3z|^2 \geq 25$, so that $|\frac{1}{4+3z}| \leq \frac{1}{5}$. Each of C_j has length π, so we get a bound for the integral of $\pi(1+\frac{1}{5})$ as desired.

Of course, you can also say that this is $\frac{1}{3}\int_C \frac{dz}{z+4/3}$, and since $-4/3$ is not inside C, the actual value of the integral is zero!

11. Following the hints,

$$\int_{|z|=1} \left(z+\tfrac{1}{z}\right)^{2n}\frac{dz}{z} = \int_{|z|=1}\left(\sum_{k=0}^{2n}\binom{2n}{k} z^k z^{-(2n-k)}\right)\frac{dz}{z}$$
$$= \sum_{k=0}^{2n}\binom{2n}{k}\int_{|z|=1} z^{2k-2n-1}\, dz.$$

The integrals of $z^{2k-2n-1}$ above all vanish, except when $k=n$, when its value is the familiar $2\pi i$. Therefore, if we then write $z = e^{i\theta}$ as suggested,

$$\binom{2n}{n}2\pi i = \int_{|z|=1}\left(z+\tfrac{1}{z}\right)^{2n}\frac{dz}{z} = \int_0^{2\pi}(2\cos\theta)^{2n} i\, d\theta.$$

Division by 2π gives the desired result.

12. The clever proof: if $f = u+iv$ is analytic, then so is $f^2 = (u^2-v^2) + i(2uv)$. Hence $\frac{1}{2}Im(z^2) = uv$ is also harmonic. The unclever proof: we have $(uv)_x = uv_x + u_xv$ and $(uv)_{xx} = u_{xx}v + 2u_xv_x + uv_{xx}$, and similarly for y, hence by the Cauchy-Riemann equations and the harmonicity of u and v,

$$(uv)_{xx} + (uv)_{yy} = u_{xx}v + 2u_xv_x + uv_{xx} + u_{yy}v + 2u_yv_y + uv_{yy}$$
$$= (u_{xx}+u_{yy})v + 2u_xv_x + 2u_yv_y + u(v_{xx}+v_{yy}) = 0\cdot v + 2(u_xv_x + (-v_xu_x)) + u\cdot 0,$$

As for u^2, any counterexample will do. The easiest one is $u(x,y) = x$, which is harmonic, but x^2 isn't.