Math 348 **Homework 5 Solutions** **Friday, Mar. 3, 2000**

Problems 1, 2 and 3 all rely on the Cauchy integral formula: if f is analytic in a domain D which contains a circle $C = |\zeta - \alpha| = \rho$ and its interior, then

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta = \frac{f^{(n)}(z)}{n!}.$$

In the examples below, $\alpha = 0$, z is implicitly given a numerical value and ζ has been renamed z.

1. Here, $\rho = 2$.

$$\frac{1}{2\pi i}\int_{|z|=2} \frac{\cos z}{z}\,dz = \cos 0 = 1 \qquad (\cos z \text{ is entire, } n = 0 \text{ and } z = 0);$$

$$\frac{1}{2\pi i}\int_{|z|=2} \frac{e^{3z}}{(z-1)^4}\,dz = \frac{\left(e^{3z}\right)'''}{3!}\Big|_{z=1} = \frac{27e^3}{6} \qquad (e^{3z} \text{ is entire, } n = 3 \text{ and } z = 1);$$

$$\frac{1}{2\pi i}\int_C e^{3z}(z-1)^4\,dz = 0 \qquad (\text{the function is entire}).$$

2. Here, $\rho = 1$.

$$\frac{1}{2\pi i}\int_C \frac{z}{e^z}\,dz = 0 \qquad (\text{the function } ze^{-z} \text{ is entire});$$

$$\frac{1}{2\pi i}\int_C \frac{e^{3z}}{(z-3)^4}\,dz = 0 \qquad (\text{the function is analytic in and on C; 3 is outside});$$

$$\frac{1}{2\pi i}\int_C \frac{dz}{3+4z} = \frac{1}{4}\frac{1}{2\pi i}\int_C \frac{dz}{z-(-3/4)}\,dz = \frac{1}{4}\cdot 1; \qquad (\ -3/4 \text{ is inside the contour}).$$

3. We first observe that

$$\frac{1}{z^2+1} = \frac{A}{z+i} + \frac{B}{z-i} \iff 1 = A(z-i) + B(z+i) \iff A = \tfrac{i}{2}, B = -\tfrac{i}{2},$$

$$\frac{1}{z(2z+1)} = \frac{A}{z} + \frac{B}{2z+1} \iff 1 = A(2z+1) + Bz \iff A = 1, B = -2.$$

Since $\pm i, 0, -\frac{1}{2}$ are within $C = \{z : |z| = 3\}$ and $\frac{1}{2z+1} = \frac{\frac{1}{2}}{z+\frac{1}{2}}$,

$$\frac{1}{2\pi i}\int_C \frac{1}{z^2+1}\,dz = \frac{1}{2\pi i}\int_C \frac{\frac{i}{2}}{z+i}\,dz + \frac{1}{2\pi i}\int_C \frac{-\frac{i}{2}}{z-i}\,dz = \tfrac{i}{2} - \tfrac{i}{2} = 0,$$

$$\frac{1}{2\pi i}\int_C \frac{1}{z(2z+1)}\,dz = \frac{1}{2\pi i}\int_C \frac{dz}{z} + \frac{1}{2\pi i}\int_C \frac{-dz}{z+\frac{1}{2}}. = 1 - 1 = 0$$

(You're probably wondering what the point of this problem was. Well, I originally had three parts, and the third had one root inside C and one outside. I cut the wrong one.)

4. Suppose $f(z) = u + iv$ is entire and $Re(f(z)) \leq M$ for all z. Let $g(z) = e^{f(z)}$, which is entire as well. Since $|e^{\alpha+i\beta}| = e^{\alpha}$, we have

$$|g(z)| = e^{Ref(z)} \leq e^{M}.$$

Liouville's Theorem now implies that g must be constant, or $g(z) = c$ for some c. This now means that for all z, $f(z)$ must take one of the values of $\log c$. Since f is entire, it is continuous, and so it can't jump from one value to another; that is, f is constant.

5. In class, I corrected the statement of this problem to indicate the correct counterclockwise fashion. I'll do these directly, with the parameterization $z(t) = e^{it}, 0 \leq t \leq \frac{\pi}{2}$. In this case, $z^2 = e^{2it}$, the principal value of $z^{1/3}$ is $e^{it/3}$ and the principal value of $Logz = it$. (One can also do these problems using a **carefully** defined antiderivative for the integrand.):

$$\int_C z^2 \, dz = \int_{t=0}^{\pi/2} e^{2it} ie^{it} dt = \frac{i}{3i} e^{3it}\Big|_0^{\pi/2} = \frac{e^{\frac{3}{2}\pi i} - e^0}{3} = -\frac{1+i}{3};$$

$$\int_C z^{1/3} \, dz = \int_{t=0}^{\pi/2} e^{it/3} ie^{it} dt = \frac{i}{\frac{4}{3}i} e^{\frac{4}{3}it}\Big|_0^{\pi/2} = \frac{e^{\frac{2}{3}\pi i} - e^0}{\frac{4}{3}} = \tfrac{3}{8}(-3 + i\sqrt{3});$$

$$\int_C zLog(z) \, dz = \int_{t=0}^{\pi/2} \left(e^{it}\right)(it)\left(ie^{it} dt\right) = \left(i\frac{t}{2} - \frac{1}{4}\right) e^{2it}\Big|_0^{\pi/2} = \frac{1}{2} - i\frac{\pi}{4}$$

In the last, we have used the following integration by parts

$$\int -te^{2it} \, dt = \frac{i}{2} \int td(e^{2it}) = \frac{i}{2} te^{2it} - \frac{i}{2} \int e^{2it} \, dt = \frac{i}{2} te^{2it} - \frac{i}{2} \frac{1}{2i} e^{2it} = \left(i\frac{t}{2} - \frac{1}{4}\right) e^{2it}$$

6. Cauchy's inequality states that if $|f(z)| \leq M$ in $|z - \alpha| \leq R$, then $|f^{(n)}(\alpha)| \leq \frac{Mn!}{R^n}$.

Suppose that f is entire and $|f(z)| \leq M|z|^m$ for large $|z|$. Pick any α and fix it, and note that, for any R, $|z - \alpha| < R$ implies $|z| < R + |\alpha|$. Then Cauchy's inequality, applied with $n = m + 1$ states that

$$|f^{(m+1)}(\alpha)| \leq \frac{M(R + |\alpha|)^m (m+1)!}{R^{m+1}}.$$

The right-hand side above goes to 0 as $R \to \infty$, hence $f^{(m+1)}(\alpha) = 0$ for all α. This means that f is a polynomial of degree at most m. Notice that we do not need the inequality on the growth of f to apply for *small* z. Otherwise, we'd have to assume that $f(0) = 0$.

7. Well, I basically did this one in class. Since e^z and $\sin z$ are both entire, they will be the uniform limit of their Taylor series on $|z| \leq \rho < \infty$ for any ρ. If we take the series at

$\alpha = 0$, then one needs to show that $f^{(n)}(0)$ is always 1 for $f(z) = e^z$ and is periodically $0, 1, 0, -1, \ldots$ for $f(z) = \sin z$. You can do that without me.

8. As before, and with n representing an arbitrary integer,

$$\sin z = \frac{5}{3} \iff \frac{e^{iz} - e^{-iz}}{2i} = \frac{5}{3} \iff e^{2iz} - \frac{10i}{3}e^{iz} - 1 = 0$$

$$\iff e^{iz} = \tfrac{1}{2}\left(\frac{10i}{3} \pm \sqrt{-\frac{100}{9} + 4}\right) = \frac{5i}{3} \pm \frac{4i}{3} \iff e^{iz} \in \{3i, i/3\} = \{3e^{i\pi/2}, \tfrac{1}{3}e^{i\pi/2}\}$$

$$\iff iz = \pm \log 3 + i\tfrac{\pi}{2} + 2n\pi i \iff z = \pm i \log 3 + \tfrac{\pi}{2} + 2n\pi$$

9. Suppose C is a contour that runs from z_0 to z_1. Then we know that $\int_C z\,dz = \frac{1}{2}(z_1^2 - z_0^2)$. Thus, $z_1 = \pm z_0$. If $z_1 = z_0$, then C is a closed contour, and we know that, provided C doesn't pass through 0, $\int_C \frac{dz}{z}$ is an integer multiple of $2\pi i$. If $z_1 = -z_0$, then as in Example 2.2, the value of the integral is the the change in a locally defined branch of the logarithm. That is, the value of the integral is $\log z_1 - \log z_0 = \log(-z_0) - \log z_0$. Since $|z_0| = |-z_0|$ and $arg(-z_0) = arg(z_0) + \pi i + 2n\pi i$, it follows that $\int_C \frac{dz}{z}$ is an odd integer multiple of πi. Putting it together, $\int_C \frac{dz}{z}$ is an integer multiple of πi.

10. This is a proof by contradiction. Suppose f is analytic for $|z| \leq 1$ and

$$\max_{|z|=1} \left| \frac{1}{z} - f(z) \right| = \lambda < 1.$$

As noted, by Cauchy's Theorem and the usual integral estimate, we have

$$2\pi i = \int_{|z|=1} \left(\frac{1}{z} - f(z)\right) \implies 2\pi \leq LM = 2\pi\lambda,$$

which is an immediate contradiction. Notice that since $\bar{z} = \frac{1}{z}$ on the unit circle, this also illustrates the difficulty of approximating $\bar{z}$ by an analytic function.

11. This is a riff on #4 above. Let $f = u + iv$ and let

$$g(z) = e^{(348-445i)f(z)} = e^{(348u+445v)+i(348v-445u)}.$$

Then $|g(z)| = e^{(348u+445v)} \leq e^{61801}$. This is an enormous, but finite bound, and it follows by Liouville's Theorem that g is constant, and as before, that f is constant.

12. I would take a branch of the log with a cut on the negative imaginary axis, so that $f(z) = \log z = \log|z| + i \arg z$, with $-\frac{\pi}{2} < \arg z \leq \frac{3\pi}{2}$. In this case,

$$\int_C \frac{dz}{z} = f(-1-i) - f(1-i) = \log\sqrt{2} + \tfrac{5}{4}\pi i - \left(\log\sqrt{2} - \tfrac{1}{4}\pi i\right) = \tfrac{3}{2}\pi i.$$