Math 348 **Homework 6 Solutions** **Thursday, Mar. 9, 2000**

1. Using the binomial theorem, and the fact that the integral of $z^n\ dz$ over the given contour is 0, if $n \neq -1$, and $2\pi i$ if $n = -1$, we have:

$$\frac{1}{2\pi i}\int_{|z|=2}\left(z+\frac{1}{z}\right)^3\ dz = \frac{1}{2\pi i}\int_{|z|=2}\left(z^3+3z+\frac{3}{z}+\frac{1}{z^3}\right)\ dz = 3.$$

One way to do he second question is to note that if $f(z) = \frac{1}{z-3}$, then f is analytic on and in the contour, hence,

$$\frac{1}{2\pi i}\int_{|z|=2}\frac{dz}{z^2-3z} = \frac{1}{2\pi i}\int_{|z|=2}\frac{f(z)}{z-0}\ dz = f(0) = -\frac{1}{3}.$$

If you do it by partial fractions, you have $\frac{1}{z^2-3z} = \frac{-1/3}{z} + \frac{1/3}{z-3}$, and need to use the fact that 3 is outside the contour.

2. Lots of different acceptable answers.

3. Using Theorem 6.3, we know that the Taylor series for an analytic function can be differentiated term by term. Hence:

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k \implies \frac{1}{(1-z)^2} = \sum_{k=0}^{\infty} kz^{k-1} = \sum_{k=1}^{\infty} kz^{k-1} = \sum_{m=0}^{\infty}(m+1)z^m.$$

Using example 6.5, that a Taylor series can be integrated term by term, with appropriate care given for the value of the constant. We are looking at a disk $|z| \leq \rho < 1$, so that $1-z$ stays in the right half plane, and we can use the Principal Value of the logarithm without fear of touching the branch cut.

$$\int_0^z \frac{1}{1-\zeta}\ d\zeta = -Log(1-\zeta)\Big|_{\zeta=0}^{z} = -Log(1-z) - Log(1) = -Log(1-z) =$$

$$\int_0^z \sum_{k=0}^{\infty}\zeta^k d\zeta = \sum_{k=0}^{\infty}\int_0^z \zeta^k d\zeta = \sum_{k=0}^{\infty}\frac{\zeta^{k+1}}{k+1}\Big|_{\zeta=0}^{z} = \sum_{k=0}^{\infty}\frac{z^k}{k+1}.$$

4. Without much explanation needed (everything converges):

$$\frac{1+2z}{(1-z)^2} = \frac{3-2(1-z)}{(1-z)^2} = \frac{3}{(1-z)^2} - \frac{2}{1-z} = \sum_{k=0}^{\infty}\left(3(k+1)-2\right)z^k = \sum_{k=0}^{\infty}(3k+1)z^k.$$

5. Using the handout from class and the hint, and the fact that $z^2+1 = (z+i)(z-i)$, we have

$$\frac{P(z)}{(z+i)(z-i)(z-1)(z-2)} = \frac{P(-i)}{6-2i}\cdot\frac{1}{z+i} + \frac{P(i)}{6-2i}\cdot\frac{1}{z-i} + \frac{P(1)}{-2}\cdot\frac{1}{z-1} + \frac{P(2)}{5}\cdot\frac{1}{z-2}$$

(If $f(z) = (z^2+1)(z-1)z-2)$, then $f'(-i) = 6-2i, f'(i) = 6+2i, f'(1) = -2, f'(2) = 5$.) Using the formula $(z+w)^{-1} = \sum_{k=0}^{\infty} -w^{-(k+1)}z^k$, derived in Monday's handout, we have then

$$\frac{P(z)}{(z+i)(z-i)(z-1)(z-2)} = \sum_{n=0}^{\infty} a_n z^n,$$

where (with my deepest and humblest apologies),

$$a_n = -\left(\frac{P(-i)}{6-2i}i^{-(n+1)} + \frac{P(i)}{6-2i}(-i)^{-(n+1)} + \frac{P(1)}{-2} + \frac{P(2)}{5}2^{-(n+1)}\right).$$

6. Let's assume that f is continuous on C and define

$$F(z) = \int_C \frac{f(\zeta)}{\zeta - z}\, dz$$

for any point z not on C. Suppose $|h| < \delta$ implies that $z+h$ is also not on z. Then by the sort of algebraic manipulations we've seen before, if $|h| < \delta/2$, then

$$\frac{F(z+h)-F(z)}{h} - \int_C \frac{f(\zeta)}{(\zeta-z)^2}\, dz =$$

$$\frac{1}{h}\left(\int_C \frac{f(\zeta)}{\zeta-(z+h)}\, dz - \int_C \frac{f(\zeta)}{\zeta-z}\, dz\right) - \int_C \frac{f(\zeta)}{(\zeta-z)^2}\, dz$$

$$= \int_C \frac{f(\zeta)}{(\zeta-z)(\zeta-(z+h))}\, dz - \int_C \frac{f(\zeta)}{(\zeta-z)^2}\, dz = h\int_C \frac{f(\zeta)}{(\zeta-z)^2(\zeta-(z+h))}\, dz.$$

Since f is continuous on C, it is bounded by M, say, and the denominator is bounded below by $\delta^2(\delta/2)$. Thus, if the length of C is denoted by L as usual,

$$\left|\frac{F(z+h)-F(z)}{h} - \int_C \frac{f(\zeta)}{(\zeta-z)^2}\, dz\right| \le h\frac{2LM}{\delta^3}.$$

As h goes to 0, this bound goes to zero, hence F is differentiable at z with the indicated derivative.

7. I'll use L'Hopital's rule on all three, since each has the form $\frac{0}{0}$. A Taylor series approach is actually easier for the last one.

$$\lim_{z\to\pi} \frac{\sin z}{\pi - z} = \lim_{z\to\pi} \frac{\cos z}{-1} = -\cos\pi = 1;$$

$$\lim_{z\to i} \frac{e^{\pi z}+1}{z^2+1} = \lim_{z\to i} \frac{\pi e^{\pi z}}{2z} = \frac{\pi(-1)}{2i} = \frac{\pi i}{2}$$

$$\lim_{z\to 0} \frac{(1-\cos z)^2}{\sin z + \sinh z - 2z} = \lim_{z\to 0} \frac{2\sin z(1-\cos z)}{\cos z + \cosh z - 2}\left(=\frac{0}{0}\right) = \lim_{z\to 0} \frac{2\cos z + 2\sin^2 z - 2\cos^2 z}{-\sin z + \sinh z}$$

$$\left(=\frac{0}{0}\right) = \lim_{z\to 0} \frac{-2\sin z + 4\sin z\cos z}{-\cos z + \cosh z}\left(=\frac{0}{0}\right) = \lim_{z\to 0} \frac{-2\cos z + 4\cos^2 z - 4\sin^2 z}{\sin z + \sinh z} = \frac{2}{0} = \infty.$$

8. As corrected in class (no discussion of singularities at ∞.) For e^z, which is entire, there are no singularities.

For $\frac{\cos z}{z}$, which is the quotient of two entire functions, the only singularity is at $z = 0$, and since $\cos 0 \neq 0$, this is a pole of order 1.

For $\frac{e^z-1}{z(z-1)}$, the only potential singularities are where the denominator vanishes, at $z = 0, 1$. The numerator has values $0, e-1$ at these points respectively. By L'Hopital's rule, we see that $\lim_{z\to 0} \frac{e^z-1}{z(z-1)} = \lim_{z\to 0} \frac{e^z}{2z-1} = -1$ exists, so $z = 0$ is a removable singularity and $z = 1$ is a pole.

The function $\frac{z^2-1}{z^2+1}$, being a quotient of polynomials, has its only singularities where the denominator vanishes, and these are at $z = \pm i$. Since the numerator doesn't vanish there, these are poles of order 1.

Finally, $\frac{z^5}{z^3+z} = \frac{z^4}{z^2+1}$ if $z \neq 0$. Thus it has poles of order 1 at $\pm i$, and a removable singularity at $z = 0$.

9. There's a little subtlety here. We proved the Schwarz lemma in class, but it doesn't quite apply, because we don't know that f is analytic to the boundary of the disk, although we do know that it's uniformly bounded. We *can* say that for every $\rho < R$, f is analytic in $|z| \leq \rho$ and is bounded by M. Thus, for every z with $|z| \leq \rho$, we have $|f(z)| \leq \frac{M}{\rho}|z|$. So now, suppose z is given with $|z| < R$, then for any ρ with $|z| < \rho < R$, we have $|f(z)| \leq \frac{M}{\rho}|z|$. If we now take the limit of this inequality as $\rho \to R$, we get the desired bound. (By the way, it is possible for a function to be analytic on $|z| < R$ and be bounded on $|z| = R$, without being analytic on the circle itself. The simplest example is something like $\sum_{n=0}^{\infty} \frac{z^n}{n^2+1}$. We'll talk more about this later.)

10. So, suppose $f = u + iv$ is analytic for $|z| < R$ and $f(0) = 0$ and suppose $u < A$ and $A > 0$. Following the hint, let

$$g(z) = \frac{f(z)}{2A - f(z)} = \frac{u + iv}{(2A - u) + iv} \implies |g(z)|^2 = \frac{u^2 + v^2}{(2A - u)^2 + v^2}.$$

Since $u < A$, it's easy to check that $u^2 < (2A - u)^2$. Since the real part of the denominator of g is positive, it's never 0, and so g is analytic, $g(0) = \frac{f(0)}{2A - f(0)} = 0$, and we can use the last problem with $M = 1$. Thus,

$$|g(z)| \leq \frac{|z|}{R} \implies \left| \frac{f(z)}{2A - f(z)} \right| \leq \frac{|z|}{R} \implies$$
$$R|f(z)| \leq |z| \cdot |2A - f(z)| \leq |z| \left(2A + |f(z)|\right) \implies (R - |z|)\, |f(z)| \leq 2A|z|.$$

11. Follow the hint and write $P(z) = \sum_{n=0}^{N} a_n (z - a)^n$. Write the contour as $z = a + Re^{it}$, $0 \leq t \leq 2\pi$. Then

$$\frac{1}{2\pi i} \int_C \overline{P(z)}\, dz = \frac{1}{2\pi i} \int_{t=0}^{2\pi} \overline{\left(\sum_{n=0}^{N} a_n R^n e^{int} \right)} iRe^{it}\, dt$$
$$= \frac{1}{2\pi i} \int_{t=0}^{2\pi} \left(\sum_{n=0}^{N} \overline{a_n} R^n e^{-int} \right) iRe^{it}\, dt = \sum_{n=0}^{N} \frac{1}{2\pi} \int_{t=0}^{2\pi} \overline{a_n} R^{n+1} e^{i(1-n)t}\, dt = R^2 \overline{a_1} = R^2 \overline{P'(0)}.$$

12. Suppose $|f(\frac{1}{n})| \leq \frac{1}{2^n}$ and f is analytic at 0. Then it is continuous at 0, and we must have $f(0) = 0$. If f is not identically zero in D, then there must exist an analytic function $g(z)$ with $g(0) \neq 0$ and a positive integer m so that $f(z) = z^m g(z)$. Since g is continuous, there exists N so that: if $|z| < 1/N$, then $|g(z)| > \frac{1}{2} g(0) = \tau > 0$ and hence $|f(z)| > \tau |z|^m$. Then, for all $n > N$, we have

$$\frac{1}{2^n} \geq \left| f\left(\frac{1}{n}\right) \right| \geq \frac{\tau}{n^m} \implies n^m \geq \tau 2^n.$$

But this inequality is impossible, because exponential functions grow more quickly than polynomials. This is a classic graduate complex exam problem.