There were an embarrassing number of small mistakes in the printed solutions to homework 6. After grading the assignment, I also found that there were some additional comments that needed to be made.

1. I didn't expect the generalized Cauchy Integral Formula (5.6) to be used here. The fastest way is to expand it out by the binomial theorem; if you're going to use the Chain Rule to differentiate $(x^2+1)^3$, use it correctly!

3. Unfortunate typo in the very last formula. Corrected version follows:

$$-Log(1-z) = \cdots = \sum_{k=0}^{\infty} \frac{\zeta^{k+1}}{k+1}\bigg|_{\zeta=0}^{z} = \sum_{k=0}^{\infty} \frac{z^{k+1}}{k+1}.$$

4. Lots of people didn't write down an explicit formula for $3k+1$. It wasn't clear in the question, but that's what I wanted.

5. This was messier than I had realized, but it's not really that difficult. One key point, and a formula that's worth remembering, is this. If $z_0 \neq 0$, then formally,

$$\frac{1}{z-z_0} = -\frac{1}{z_0}\cdot\frac{1}{1-z/z_0} = -\frac{1}{z_0}\sum_{n=0}^{\infty}\left(\frac{z}{z_0}\right)^n = -\sum_{n=0}^{\infty}\frac{z^n}{z_0^{n+1}}.$$

The geometric series converges provided $|z/z_0| < 1$; that is, provided $|z| < |z_0|$, so this expression provides the correct Taylor series at the origin. I made several computational errors in the solution, so here it is, corrected:

$$\frac{P(z)}{(z+i)(z-i)(z-1)(z-2)} = \frac{P(-i)}{6-2i}\cdot\frac{1}{z+i} + \frac{P(i)}{6+2i}\cdot\frac{1}{z-i} + \frac{P(1)}{-2}\cdot\frac{1}{z-1} + \frac{P(2)}{5}\cdot\frac{1}{z-2},$$

so $\frac{P(z)}{(z+i)(z-i)(z-1)(z-2)} = \sum_{n=0}^{\infty} a_n z^n$ implies

$$\begin{aligned} a_n &= -\left(\frac{P(-i)}{6-2i}(-i)^{-(n+1)} + \frac{P(i)}{6+2i}(i)^{-(n+1)} + \frac{P(1)}{-2} + \frac{P(2)}{5}2^{-(n+1)}\right) \\ &= \frac{P(-i)}{2+6i}i^n + \frac{P(i)}{2-6i}(-i)^n + \frac{P(1)}{2} - \frac{P(2)}{10}\frac{1}{2^n}. \end{aligned}$$

6. The mathematical point of this problem is that you can't blithely say that $\lim_{n\to\infty}\int_C f_n(z)\ dz = \int_C \lim_{n\to\infty} f_n(z)\ dz$ without good cause, like a theorem! This problem was an instance of this sort of theorem, and it was quite do-able in the context of p. 135 or class discussion.

7. If L'Hospital's (or L'Hôpital's) rule has to be used more than once, the smart money is on writing the functions as Taylor series. In the last instance, we have

$$\begin{aligned} (1-\cos z)^2 &= \left(1-\left(1-\frac{z^2}{2!}+\ldots\right)\right)^2 = \frac{z^4}{(2!)^2}+\ldots \\ \sin z + \sinh z - 2z &= \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots\right) + \left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots\right) - 2z = \frac{2z^5}{5!} + \ldots, \end{aligned}$$

so the numerator has a zero of degree 4 and the denominator has a zero of degree 5, and the quotient actually has a pole of order 1 at $z=0$. If you are going to differentiate, recall that $(\cosh z)' = \sinh z$; this is one salient way the hyperbolic trig functions are different.

8. Explanations are always needed, and "pole" isn't a sufficient description. Give the order.

9. Read the proof. This one is subtle, and you have to know how to apply the theorems you know.

10, 11. Don't worry too much about these.

12. Most of the proofs here were wrong, and you could have checked them by noting that all they used was that $2^{-n} \to 0$, and that they would have worked equally well with the condition $|f(\frac{1}{n})| = \frac{1}{n}$. But $f(z) = z$, which isn't the 0 function, satisfies this. The idea was that $2^{-n} \to 0$ too quickly. One final point: this problem depends critically on the assumption that f is analytic at $z=0$. If $f(z) = e^{-z^{-2}}$, then

$$f(\tfrac{1}{n}) = e^{-n^2} = \frac{1}{(e^n)^n} < \frac{1}{2^n}.$$

Since f has an essential singularity at $z=0$, this is not a violation of the problem. (And look at $f(i\epsilon)$ to see a function growing rather rapidly!)