1. A series at $z = 1$ is a series *in* $z - 1$, so with the usual manipulations, we have:

$$\frac{1}{2z+3} = \frac{1}{2(z-1)+5} = \frac{\frac{1}{5}}{1 - \frac{-2}{5}(z-1)} = \frac{1}{5}\sum_{n=0}^{\infty}\left(-\frac{2}{5}\right)^n (z-1)^n = \sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{5^{n+1}}(z-1)^n.$$

This can also be done by successively differentiating $f(z) = \frac{1}{2z+3} = \frac{1}{2}\cdot\frac{1}{z+\frac{3}{2}}$. We have, by an easy induction,

$$f^{(n)}(z) = \frac{1}{2}\cdot\frac{(-1)^n n!}{(z+\frac{3}{2})^{n+1}} \implies \frac{f^{(n)}(1)}{n!} = \frac{1}{2}\cdot\frac{(-1)^n 2^{n+1}}{5^{n+1}}.$$

2., 3. In part, this is a sneaky question because $\frac{1}{z+2}$ is already a Laurent series in $z+2$ (!) which converges when $|z+2| > 0$. We express the other summand in terms of $z = 2$:

$$\frac{1}{1-2z} = \frac{1}{5-2(z+2)}.$$

The natural thing to do is to factor out the "5" as a geometric series to obtain:

$$\frac{1}{5-2(z+2)} = \frac{1}{5}\cdot\frac{1}{1-\frac{2(z+2)}{5}} = \sum_{n=0}^{\infty}\frac{2^n}{5^{n+1}}(z+2)^n,$$

very much as we did in 1. Where does this converge? We need $|\frac{2(z+2)}{5}| < 1$, that is, $|z+2| < \frac{5}{2}$. Combined with the other part, we can now answer question 3 first: The Laurent series which converges in $0 < |z+2| < \frac{5}{2}$ is

$$f(z) = \frac{1}{z+2} + \sum_{n=0}^{\infty}\frac{2^n}{5^{n+1}}(z+2)^n.$$

What are we going to do if $|z+2| > \frac{5}{2}$? Well this is the same thing as saying that $|\frac{5}{2(z+2)}| < 1$, so this just requires a little symbolic manipulation:

$$\frac{1}{5-2(z+2)} = \frac{1}{-2(z+2)}\cdot\frac{1}{1-\frac{5}{2(z+2)}}$$

This last series is another geometric one, and $|\frac{5}{2(z+2)}| < 1$ in this case, so we have

$$\frac{1}{-2(z+2)}\cdot\frac{1}{1-\frac{5}{2(z+2)}} = \frac{1}{-2(z+2)}\sum_{n=0}^{\infty}\left(\frac{5}{2(z+2)}\right)^n = -\sum_{n=0}^{\infty}\frac{5^n}{2^{n+1}}\frac{1}{(z+2)^{n+1}},$$

and, combining it with the first term, we obtain an expression whose convergence is obvious when $|z+2|$ is large:

$$f(z) = \frac{1}{z+2} - \sum_{n=0}^{\infty} \frac{5^n}{2^{n+1}} \frac{1}{(z+2)^{n+1}} = \left(1 - \frac{5^0}{2^1}\right) \cdot \frac{1}{z+2} - \sum_{n=1}^{\infty} \frac{5^n}{2^{n+1}} \frac{1}{(z+2)^{n+1}}$$

4. (a) It's clear that e^z has an essential singularity at ∞, because it goes to ∞ along the positive real axis, but is bounded by 1 in absolute value on the imaginary axis. This behavior is impossible in a pole or a removable singularity. (b) For similar reasons, $\frac{\cos z}{z}$ also has an essential singularity at ∞: it is bounded for large real values of z, but is unbounded for $z = iy$. (c) Similarly again, $\frac{e^z-1}{z(z-1)}$ is bounded on the imaginary axis and goes to ∞ on the positive real axis. Also essential. (d) On the other hand, $\frac{z^2+1}{z^2-1}$ is bounded as $z \to \infty$, so that ∞ is a removable singularity. (e) Finally, $f(z) = \frac{z^5}{z^3+z} = \frac{z^4}{z^2+1}$ (for $z \neq 0$) goes to ∞ as $z \to \infty$, so $z = \infty$ is a pole. To determine the order, we take $g(z) = f(\frac{1}{z}) = \frac{(1/z)^4}{(1/z)^2+1} = \frac{1}{z^2+z^4} = \frac{\frac{1}{1+z^2}}{z^2}$, which has a pole of order 2 at $z = 0$, hence f has a pole of order 2 at $z = \infty$. It's also OK to write these functions in Laurent series at ∞, and use Theorem 9.4.

5. Well $f(z)$ is analytic when $e^{z^2} - 1$ is analytic and not equal to zero. This function is entire, so we need only look at those places where $e^{z^2} = 1$, that is, where $z^2 = \pm 2n\pi i$ (for integers $n \geq 0$), and at $z = \infty$. It should be no shock that $z = \infty$ is an essential singularity, because $e^{z^2} - 1 \to \infty$ for real z, hence $f \to 0$ is bounded on the real axis, but $e^{(iy)^2} - 1 = e^{-y^2} - 1 \to -1$, so $f \to -1$ on the imaginary axis.

Otherwise, $f(z)$ has a pole at any point z_0 where $e^{z_0^2} = 1$, and the order of the pole is equal to the order of the zero of $g(z) = e^{z^2} - 1$ at $z = z_0$. Observe that $g'(z) = 2ze^{z^2}$, so if $z_0 \neq 0$, then $g'(z_0) = 2z_0 e^{z_0^2} = 2z_0 \neq 0$, so that g has a zero of order 1 and f has a pole of order 1. These points are the various square roots of $\pm 2n\pi i$ for $n \geq 1$ and can be written as $\sqrt{n\pi}(\pm_1 \pm_2 i)$, for all four choices of sign. If $z_0 = 0$, then $e^{z^2} - 1 = \sum_{n=0}^{\infty} \frac{(z^2)^n}{n!} - 1 = z^2 + \ldots$ has a zero of order 2 at $z = 0$, so that f has a pole of order 2 at $z = 0$.

6. We have

$$f(z) = \frac{(z-2)^2 \sin \frac{1}{z}}{z^3 - 4z} = \frac{(z-2)^2 \sin \frac{1}{z}}{z(z-2)(z+2)},$$

so the denominator implies that there are isolated singularities at $z = 0, 2, -2$ and the numerator indicts $z = 0$; $z = \infty$ is always a suspect. As $z \to 0$, we see that $\frac{(z-2)^2}{z^3-4z}$ has a pole of order 1, but $\sin \frac{1}{z}$ has an essential singularity, and that carries the day. It's easy to see that $z = 2$ is a removable singularity, because of the factor $(z-2)^2$ in the numerator. Finally, $z = -2$ is a pole of order 1, because everything else is well-behaved and not zero as $z \to 2$. To understand $z = \infty$, we take

$$g(z) = f(\tfrac{1}{z}) = \frac{(\frac{1}{z} - 2)^2 \sin z}{(\frac{1}{z})^3 - 4\frac{1}{z}} = \frac{z(1-2z)^2 \sin z}{1 - 4z^2}.$$

Since g is bounded near $z = 0$, it follows that $z = \infty$ is a removable singularity for f.

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8. So f and g are entire functions. It follows that $h = \frac{f}{g}$ is analytic except at z_0, where $g(z_0) = 0$. Since g is not identically zero, its zeros are isolated, and for each zero, there is a punctured neighborhood on which g is not zero. That is, there exists r (which depends on z_0) so that $g(z) \neq 0$ for $0 < |z - z_0| < r$. It follows that h has an isolated singularity in this neighborhood. But $|h(z)| \leq 2$ wherever it's defined. It follows that h has a removable singularity at z_0, and it can be "filled in" with γ so that $|\gamma| \leq 2$.

Now define $H(z)$ to equal $h(z)$ where h is defined, and the filled-in values at the zeros of g. Then H is an entire function and $|H(z)| \leq 2$ for all z. It follows by Liouville's Theorem that there exists a constant c so that $H(z) = c$. Now, if $g(z_0) = 0$, then $|f(z_0)| \leq 2|g(z_0)| = 0$, so $f(z_0) = 0$, and if $g(z_0) \neq 0$, then $f(z_0) = cg(z_0)$; hence $f(z) = cg(z)$ for all z.

9. These are mainly formal manipulations. We have

$$\frac{e^z}{z^5} = \frac{1}{z^5} \sum_{n=0}^{\infty} \frac{z^n}{n!} = \sum_{m=-5}^{\infty} \frac{z^m}{(m+5)!},$$

so the principal part is $\frac{1}{z^5} + \frac{1}{z^4} + \frac{1}{2z^3} + \frac{1}{6z^2} + \frac{1}{24z}$. For the second, note that $\sin z = \sin(z - 2\pi)$, hence

$$\frac{\sin z}{(z-2\pi)^2} = \frac{\sin(z-2\pi)}{(z-2\pi)^2} = \frac{1}{(z-2\pi)^2} \cdot \sum_{k=0}^{\infty} (-1)^k \frac{(z-2\pi)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} (-1)^k \frac{(z-2\pi)^{2k-1}}{(2k+1)!}$$

with principal part equal to the first term only, $\frac{1}{z-2\pi}$.

10. Following the extremely unsubtle hint, we have from the handout that

$$|z| \geq 1 \implies |p(z)| \geq |z|^n - (|a_0| + \cdots + |a_{n-1}|)|z|^{n-1} = |z|^n \left(1 - \frac{(|a_0| + \cdots + |a_{n-1}|)}{|z|}\right).$$

If we choose $R = 1 + 100(|a_0| + \cdots + |a_{n-1}|)$, then if $|z| \geq R$, we have $|z| \geq 1$ and $\frac{(|a_0| + \cdots + |a_{n-1}|)}{|z|} \leq .01$, which is what we want.

12. Suppose f is analytic in $|z| \leq 1$ and $f(\alpha) = 0$ for some α with $|\alpha| < 1/$ Observe that $w = \frac{z-\alpha}{1-\overline{\alpha}z}$ is equivalent to $z = \frac{w+\alpha}{1+\overline{\alpha}w}$. Accordingly, let

$$f(z) = g\left(\frac{z-\alpha}{1-\overline{\alpha}z}\right) \iff g(z) = f\left(\frac{z+\alpha}{1+\overline{\alpha}z}\right)$$

The hint is an example which shows that when $|\alpha| < 1$, we have $|z| \leq 1$ if and only if $|\frac{z-\alpha}{1-\overline{\alpha}z}| \leq 1$. Thus, g is also analytic in $|z| \leq 1$. Since $f(\alpha) = g(0) = 0$, Schwarz's Lemma implies that $|g(z)| \leq |z|$ inside the unit circle, hence $|f(z)| = |g(\frac{z-\alpha}{1-\overline{\alpha}z})| \leq |\frac{z-\alpha}{1-\overline{\alpha}z}|$.