Math 348 Homework 7 Corrections and Amplifications 3/27/00

First, a correction on problem 5. If $f(z) = \frac{1}{e^{z^2}-1}$, then f has an isolated singularity at every zero of $e^{z^2} - 1$. But

$$e^{z^2} - 1 = 0 \iff e^{z^2} = 1 \iff z^2 = 2n\pi i,$$

and so there are infinitely many isolated singularities, and these are unbounded. This means that ∞ is not an isolated singularity. (Recall that, by definition, a neighborhood of ∞ is a set of the form $|z| > R$; that is, the exterior of a closed disk.) So if ∞ is an isolated singularity, then all "finite" singularities must be confined to $|z| \le R$ for some R. Incidentally, this is half of an important theorem: If f is analytic in the whole complex plane, except for infinitely many isolated singularities, then it has a non-isolated singularity at ∞. The proof is that, suppose ∞ is an isolated singularity: the argument above shows that f must have all its singularities in $|z| \le R$ for some R. But an infinite set of points (the singularities) in a compact set ($|z| \le R$) must have an accumulation point, say z_0, and z_0 would not then be an isolated singularity. The reason it's useful to look at ∞ is that it "compactifies" the complex plane and allows us to treat the complex plane as if it were a compact set, as in this case.

On Taylor series and Laurent series. If $f(z)$ is analytic in $|z - \alpha| < r$ for some $r > 0$, then we know that f can be expressed as a Taylor series, and

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(z-\alpha)^n$$

in $|z - \alpha| < r$, with convergence uniform is $|z - \alpha| < \rho$ for any $\rho < r$. If f is not analytic at α, then a series like this is only possible if f has a removable singularity at α: in this case, we let g denote f with the missing value filled in, and then expand f as a Tayor series. If f has a pole of order m at α, we can find the Laurent series by writing $f(z) = \frac{g(z)}{(z-\alpha)^m}$, and then expanding g as a Taylor series at $z = \alpha$. If f has an essential singularity, then we're in trouble, and we can only hope to find some other way to express it.

On the classification of Laurent series at ∞. Recall that, to analyze $f(z)$ with an isolated singularity at $z = \infty$, we look at $g(z) = f(1/z)$ at $z = 0$. We can classify the singularity of g at 0 by looking at its Laurent series at $z = 0$, and then translate the information back to f at ∞:

$$\begin{aligned}
g(z) &= \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{z^n}, &&\text{convergent for } |z| < r \\
\Longrightarrow\ f(\tfrac{1}{z}) &= \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{z^n}, &&\text{convergent for } |1/z| < r \\
\Longrightarrow\ f(z) &= \sum_{n=0}^{\infty} \frac{a_n}{z^n} + \sum_{n=1}^{\infty} a_{-n} z^n &&\text{convergent for } |z| > 1/r.
\end{aligned}$$

Thus, if g is analytic at $z = 0$, then we can say that f is analytic at $z = \infty$ and has a "Taylor series at ∞" of the form $f(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}$. This is an ordinary power series in the variable $1/z$. The same can be said if g has a removable singularity at $z = \infty$. This is how you spot a removable singularity at ∞: if the function is analytic for $|z| \ge R$ and converges to some number a_0 as $|z| \to \infty$, then ∞ is a removable singularity.

If g has a pole of order m at 0, then f has a series at ∞ of the form

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n} + a_{-1}z + \cdots + a_{-m}z^m,$$

with $a_{-m} \neq 0$. In other words, if $f(z) - p(z) \to 0$ as $|z| \to \infty$ for some polynomial p of degree m, then f has a pole at ∞ of order m.

If g has an essential singularity at 0, then f has a series at ∞ with infinitely many terms with positive exponent. Thus, if f is entire and not a polynomial, then it must have an essential singularity at ∞ for this reason.

But what about $f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, which has infinitely many terms with positive exponent, and which we know has a removable singularity at $z = \infty$? The key here is that $\sum_{n=0}^{\infty} z^n$ is only convergent for $|z| < 1$, and so is **not** the Laurent series for f at ∞.

Suppose we are interested in expanding $f(z) = \frac{1}{z-\alpha}$ in a series at $z = z_0$. The first thing to do is to write $z - \alpha = z - z_0 - (\alpha - z_0)$, in order to make the variable $z - z_0$. **I will assume that $z_0 = 0$ for simplicity in the rest of this paragraph, but in practice, you have to make the translation of variable noted above!** So you want to write $f(z) = \frac{1}{z-\alpha}$ as a series in z. If $\alpha = 0$, then f is not analytic at $z = 0$, but is already a Laurent series that converges for all z with $|z| > 0$. If $\alpha \neq 0$, then f is analytic in the disk $|z| < |\alpha|$ and so has a Taylor series that converges there. With familiar manipulations,

$$\frac{1}{z-\alpha} = \frac{1}{-\alpha} \cdot \frac{1}{1 - \frac{z}{\alpha}} = -\sum_{n=0}^{\infty} \frac{z^n}{\alpha^{n+1}}.$$

This series is convergent for $|z| < |\alpha|$ by substitution into the geometric series. What's new is that we can talk about a Laurent series which converges outside the disk; that is, in $|z| > |\alpha|$. This series is nothing more than the Taylor series at ∞, which is OK since f only has the single point α where it isn't analytic. We get, by other familiar manipulations:

$$\frac{1}{z-\alpha} = \frac{1}{z} \cdot \frac{1}{1 - \frac{\alpha}{z}} = \sum_{n=0}^{\infty} \frac{\alpha^n}{z^{n+1}}.$$

This series is convergent for $|z| > |\alpha|$. That's all there is to it.

It is understandable, considering the lack of discussion in the book or class, to try to find these coefficients via the integral formulas. One thing you have to keep in mind is that the Cauchy Integral Formula (5.6):

$$\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

only applies for integers $n \geq 0$. This is not a profitable way to evaluate the integral for negative n unless you want to change your definitions! Nonetheless, it is possible to do #3 in this way. We wanted the Laurent series for $\frac{1}{1-2z}$ which converges in $0 < |z+2| < \frac{5}{2}$. Theorem 9.2 certainly states that the coefficient a_j is given by

$$\frac{1}{2\pi i} \int_{|\zeta+2|=c} \frac{\frac{1}{1-2\zeta}}{(\zeta+2)^{j+1}}\, d\zeta.$$

For $j \geq 0$, this can be evaluated as before to get the Taylor series, and if $j \leq -1$, then the integrand is analytic in $|\zeta + 2| \leq c$, so it's equal to zero.