Math 348 **Homework 8 Solutions** **Friday, Mar. 31, 2000**

---

1. A Laurent series in z has the shape $\sum_{n=0}^{\infty} a_n(z-\alpha)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-\alpha)^n}$. Thus,

$$f(z) = \frac{z-2}{z} = 1 - \frac{2}{z}$$

is already a Laurent series for $\alpha = 0$. It converges for $|z| > 0$. To write the Tayor series for f at $\alpha = 1$, we do a little algebraic manipulation to express f in terms of $z-1$:

$$f(z) = \frac{z-2}{z} = \frac{(z-1)-1}{(z-1)+1} = (z-1)\frac{1}{(z-1)+1} - \frac{1}{(z-1)+1}.$$

This Taylor series is supposed to converge in a neighborhood of 1, and the singularity of f is at $z = 0$, a distance of 1. Thus, we are seeking convergence in $|z+1| < 1$, and this dictates the following series:

$$\begin{aligned} f(z) = (z-1)\frac{1}{1+(z-1)} - \frac{1}{1+(z-1)} &= (z-1)\sum_{n=0}^{\infty}(-1)^n(z-1)^n - \sum_{n=0}^{\infty}(-1)^n(z-1)^n \\ = \sum_{n=0}^{\infty}(-1)^n(z-1)^{n+1} - \sum_{n=0}^{\infty}(-1)^n(z-1)^n &= -1 + 2\sum_{n=1}^{\infty}(-1)^{n+1}(z-1)^n. \end{aligned}$$

2. Sorry for the ambiguity in the book. The functions $\frac{\sin z}{z}$, $\frac{e^z-1}{z}$, $\frac{Log(1+z)}{z}$ all have removable singularities at $z = 0$, and when they are defined to equal 1 there, the resulting functions are analytic at 0. (In addition to *near* 0.) Using the manipulation of the familiar power series, and the fact that we can integrate these series term by term, we have:

$$\begin{aligned} \int_0^z \frac{\sin\zeta}{\zeta}\,d\zeta &= \int_0^z (1 - \tfrac{1}{6}\zeta^2 + \tfrac{1}{120}\zeta^4 + \dots)\,d\zeta = z - \tfrac{1}{18}z^3 + \tfrac{1}{600}z^5 + \cdots = \sum_{n=0}^{\infty} \tfrac{(-1)^n z^{2n+1}}{(2n+1)!(2n+1)}; \\ \int_0^z \frac{e^\zeta - 1}{\zeta}\,d\zeta &= \int_0^z (1 + \tfrac{1}{2}\zeta + \tfrac{1}{6}\zeta^2 + \dots)\,d\zeta = z + \tfrac{1}{4}z^2 + \tfrac{1}{18}z^3 + \cdots = \sum_{n=1}^{\infty} \tfrac{1}{n*n!}z^n; \\ \int_0^z \frac{Log(1+\zeta)}{\zeta}\,d\zeta &= \int_0^z (1 - \tfrac{1}{2}\zeta + \tfrac{1}{3}\zeta^2 + \dots)\,d\zeta = z - \tfrac{1}{4}z^2 + \tfrac{1}{9}z^3 + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}z^n. \end{aligned}$$

Some books note that $m * m! = (m+1)! - m!$.

3. This problem calls for finding the principal part of $f(z) = \frac{8z^3}{(z+1)(z-1)^2}$ at $z = -1$ and $z = 1$, where f evidently has poles of order 1 and 2 respectively. If $g(z) = \frac{h(z)}{(z-\alpha)^m}$, where h is analytic at α, then the principal part of g at α is just the initial portion of the Taylor series for h at α up to the term $(z-\alpha)^{m-1}$ multiplied by $(z-\alpha)^{-m}$.

For $\alpha = -1$, we write $f(z) = \frac{h(z)}{z+1}$, where $h(z) = \frac{8z^3}{(z-1)^2}$ and $h(-1) = \frac{-8}{4} = -2$, so the principal part is $\frac{-2}{z+1}$. For $\alpha = 1$, we write $f(z) = \frac{h(z)}{(z-1)^2}$, where $h(z) = \frac{8z^3}{z+1}$. Since $h(1) = \frac{8}{2} = 4$ and by the quotient rule, $h'(1) = \frac{2*24-8*1}{2^2} = 10$, the Taylor series for h at $z = 1$ begins $h(z) = 4 + 10(z-1) + \dots$, and the principal part of f is $\frac{4}{(z-1)^2} + \frac{10}{z-1}$, as claimed.

4. Following my hint, and with a suspicious similarity to #1,

$$\frac{z-1}{z+1} = \frac{z}{z+1} - \frac{1}{z+1} = \frac{1}{1+1/z} - \frac{1/z}{1+1/z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^n} - \frac{1}{z}\sum_{n=0}^{\infty} \frac{(-1)^n}{z^n} = 1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{z^n}.$$

5. Writing out the power series far enough (at least to z^{11} in the numerator), we have

$$\frac{\sin(z^2) - z^2}{z^{11}} = \frac{z^2 - \frac{1}{3!}z^6 + \frac{1}{5!}z^{10} - \frac{1}{7!}z^{14} + \cdots - z^2}{z^{11}} = -\frac{1}{6}\cdot\frac{1}{z^5} + \frac{1}{120}\cdot\frac{1}{z} - \frac{1}{5040}z^3 + \cdots.$$

Thus, this function has a pole of order 5 at $z = 0$, and its residue is $\frac{1}{120}$.

6. By taking the reciprocal, we want to make sure that the denominator has a zero of order 4 at $z = 0$, which means that the coefficients of $1, z, z^2, z^3$ must vanish. But

$$\frac{1}{1+z} - a\cos(bz) - c\sin(dz) = 1 - z + z^2 - z^3 - a\left(1 - \frac{b^2}{2}z^2\right) - c\left(dz - \frac{d^3}{6}z^3\right) + \ldots$$

We obtain four equations:

$$1 - a = 0; \qquad -1 - cd = 0; \qquad 1 + \frac{ab^2}{2} = 0; \qquad -1 + \frac{cd^3}{6} = 0.$$

The first and third combine to $a = 1$ and $b^2 = -2$, so $b = \pm\sqrt{2}i$. (The sign doesn't matter because cos is even.) From the second and fourth, we have $cd = -1$ and $cd^3 = -6$, which combine to $(c, d) = \pm(\frac{i}{\sqrt{6}}, i\sqrt{6}$. (The signs don't matter because $-c\sin(-dz) = c\sin dz$.) Taking account of the imaginary numbers, we see that the good approximation to $\frac{1}{1+z}$ is actually $\cosh(\sqrt{2}z) - \frac{1}{\sqrt{6}}\sinh(\sqrt{6}z)$.

7. Suppose f has an isolated singularity at $z = \alpha$. If it is removable, then it's easy to see that e^f will also have a removable singularity at $z = \alpha$. If the singularity is essential, then it's easy to believe that e^f will have an essential singularity at α as well. Here's a proof by Caseroti-Weierstrass: given any complex number $z_0 \neq 0$, pick a value $w = \log z_0$. Then by C-W, $f(z)$ takes values arbitrarily close to w for z arbitrarily close to α, hence $e^{f(z)}$ takes values arbitrarily close to z_0. The remaining case is where f has a pole at $z = \alpha$, and we must show that e^f cannot also have a pole.

There may be other proofs than this one, but this seems the most natural to me. Suppose $f = u + iv$ and e^f both have poles at $z = \alpha$. Then $|e^f| = e^u$ must be large near α, which means that u is large and positive. We'll show that this is a contradiction. Suppose f has a pole of order m at $z = \alpha$, and $f(z) = g(z)(z-\alpha)^{-m}$, where $g(\alpha) \neq 0$. Suppose $g(\alpha) = \rho e^{i\theta}$, Since g is analytic at α, it is continuous there, and so there exists η so that for $|z - \alpha| < \eta$, the argument of $g(z)$ is in the interval $(\theta - \pi/4, \theta + \pi/4)$. (Nothing special about $\pi/4$ here.). Now consider complex numbers near α: $z = \alpha + \epsilon e^{it}$, where ϵ is small. We have

$$f(\alpha + \epsilon e^{it}) = \frac{g(\alpha + \epsilon e^{it})}{\epsilon^m e^{imt}} = \frac{|g(\alpha + \epsilon e^{it})|}{\epsilon^m} e^{(arg(g(\alpha+\epsilon e^{it}))-mt)i}$$

Thus, if we choose $t = -(\frac{\pi-\theta}{m})$, then the argument of $f(\alpha + \epsilon e^{it})$ is

$$arg(g(\alpha + \epsilon e^{it})) - \theta + \pi \in \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right),$$

and so in particular, $f(\alpha+\epsilon e^{it})$ has negative real part. This means that $|e^{f(z)}|<1$ along a line as $z\to\alpha$, and so e^f can't have a pole there. This is a fancy way of saying that if f has a pole at α, then the image of any punctured neighborhood of α is a neighborhood of ∞.

8. Let $f(z)=\frac{1}{z^2}(z-2)$ and let r_0 denote the residue of f at $z=0$ and r_2 denote the residue of f at $z=2$. Then the given four integrals will be $2\pi i r_0$, $2\pi i(r_0+r_2)$, $-2\pi i r_2$ and 0 respectively. We have

$$\frac{1}{z^2(z-2)}=\frac{(z-2)^{-1}}{z^2}=\frac{-\frac{1}{2}-\frac{z}{4}-\frac{z^2}{8}+\dots}{z^2}=-\frac{1}{2}\cdot\frac{1}{z^2}-\frac{1}{4}\cdot\frac{1}{z}-\frac{1}{8}+\dots;$$
$$\frac{1}{z^2(z-2)}=\frac{z^{-2}}{z-2}=\frac{(2+z-2)^{-2}}{z-2}=\frac{1}{4}\cdot\frac{1}{z-2}+\dots.$$

Thus $r_0=-\frac{1}{4}$, $r_2=\frac{1}{4}$, and the answers are $-\frac{\pi i}{2}$, 0, $-\frac{\pi i}{2}$ and 0 respectively. It is not an accident that the sum of the residues is 0. Stay tuned.

9. Note here that e^{4z} is entire. If ζ is outside a simple closed contour C, then the integrand is analytic and $g(z)=0$. If ζ is inside C, then the integrand has a singularity at $\zeta=z$, a pole of order four. Hence either by taking the residue at z, or, more directly, by using (5.6), we see that, for z inside C,

$$g(z)=2\pi i\frac{(e^{4\zeta})'''(z)}{3!}=2\pi i\frac{4^3e^{4z}}{6}=\frac{64i}{3}e^{4z}.$$

10. Because the Taylor series for e^z converges for all complex z, we may substitute into it and get a series that converges wherever it's defined. Thus, the Laurent series for e^{1/z^2} at $z=0$ is $e^{1/z^2}=\sum_{n=0}^{\infty}\frac{(1/z^2)^n}{n!}=\sum_{n=0}^{\infty}\frac{1}{n!}\cdot\frac{1}{z^{2n}}$. There are infinitely many non-zero terms in this Laurent series and the coefficient of $\frac{1}{z}$ is 0, hence the residue is 0.

11a. $f(z)=\frac{1}{z(z-1)}$ has simple poles at $z=0,1$, and residues equal to -1 and 1 respectively, because $f(z)=\frac{1/(z-1)}{z}=\frac{1/z}{z-1}$

11b. $f(z)=\frac{z}{z^4+1}$ has simple poles at $z=e^{i\pi/4},e^{3i\pi/4},e^{5i\pi/4},e^{7i\pi/4}$, and the residue at any pole z_0 is $\frac{z_0}{4z_0^3}=\frac{1}{4z_0^2}$ by (2.4). Thus, the residues are $-\frac{i}{4},\frac{i}{4},-\frac{i}{4},\frac{i}{4}$, respectively.

11c. $f(z)=\frac{\sin z}{z^2(\pi-z)}$ has singularities only at $z=0,\pi$. Writing out the Taylor series of the numerator and denominator, we find:

$$\frac{\sin z}{z^2(\pi-z)}=\frac{z-\frac{1}{6}z^3+\dots}{\pi z^2-z^3}=\frac{1-\frac{1}{6}z^2+\dots}{\pi z-z^2}$$

hence the residue at $z=0$ is $\frac{1}{\pi}$. It's a little easier at $z=\pi$: we have $f(z)=\frac{-\sin z/z^2}{z-\pi}$, so the residue is $-\sin\pi/\pi^2=0$. In other words, f has a removable singularity at $z=\pi$.

12. In the first case, the singularities are at $z_k=\pm\frac{i}{2}$, and the residue is $\frac{e^{\pi z_k}}{8z_k}$, so the total is

$$2\pi i\left(\frac{e^{\pi i/2}}{4i}+\frac{e^{-\pi i/2}}{-4i}\right)=2\pi i\left(\frac{1}{4}+\frac{1}{4}\right)=\pi i.$$

In the second case, the singularity is obviously at $z=0$ and since $\frac{e^z}{z^3}=\frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{2}\cdot\frac{1}{z}+\dots$, the residue is $1/2$, and the integral is the same. If we change to $|z|=2$, nothing changes because all the residues in the complex plane can be found within $|z|=1$.