1., 2. The contour is the usual protractor, with function $f(z) = \frac{z^2}{(z^2+9)^3}$, which has poles of order 3 at $\pm 3i$. Since f is a rational function, and the denominator has degree at least two more than the denominator of the numerator, the integral over the half-circle goes to 0 as $R \to \infty$. Accordingly, if we let $g(z) = \frac{z^2}{(z+3i)^3}$ so that $f(z) = \frac{g(z)}{(z-3i)^3}$, then we have

$$\int_{C_1,R} \frac{z^2}{(z^2+9)^3}\,dz + \int_{C_2,R} \frac{z^2}{(z^2+9)^3}\,dz = \int_{C,R} \frac{z^2}{(z^2+9)^3}\,dz = 2\pi i Res(f(z),3i) \implies$$

$$\int_{-R}^{R} \frac{x^2}{(x^2+9)^3}\,dx + o(1) = 2\pi i \frac{g''(3i)}{2!} \implies \int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)^3}\,dx = \pi i g''(3i).$$

Since $g(z) = z^2(z+3i)^{-3} \implies g''(z) = 2(z+3i)^{-3} - 12z(z+3i)^{-4} + 12z^2(z+3i)^{-5}$, we have $g''(3i) = 2(6i)^{-3} - 36i(6i)^{-4} + 108i^2(6i)^{-5} = \frac{2-6+3}{216i^3} = -\frac{i}{216}$, we have

$$\int_0^{\infty} \frac{x^2}{(x^2+9)^3}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)^3}\,dx = \frac{1}{2}\pi i\left(-\frac{i}{216}\right) = \frac{\pi}{432}.$$

3., 4. As we've seen, there is an immediate way to translate trigonometric integrals on $[0,\pi]$ into contour integrals on $|z|=1$. In fact, by the formula on the top of p. 193,

$$\int_0^{2\pi} \frac{\cos\theta}{5+3\cos\theta}\,d\theta = \frac{1}{i}\int_{|z|=1} \frac{(z^2+1)/2z}{5+3(z^2+1)/2z}\frac{dz}{z}.$$

A little algebraic simplification turns this into

$$2\pi\frac{1}{2\pi i}\int_{|z|=1} \frac{1}{z}\frac{z^2+1}{3z^2+10z+3}\,dz = 2\pi\frac{1}{2\pi i}\int_{|z|=1} \frac{z^2+1}{z(3z+1)(z+3)}\,dz$$

The integrand has simple poles at 0, $-\frac{1}{3}$, -3, and so by the residue theorem, with $f(z) = \frac{z^2+1}{3z(z+\frac{1}{3})(z+3)}$,

$$\int_0^{2\pi} \frac{\cos\theta}{5+3\cos\theta}\,d\theta = 2\pi\left(Res(f(z),0) + Res(f(z),-\tfrac{1}{3})\right)$$

$$= 2\pi\left(\frac{0^2+1}{3\cdot\frac{1}{3}\cdot 3} + \frac{(\frac{-1}{3})^2+1}{3\cdot\frac{-1}{3}\cdot(3-\frac{1}{3})}\right) = 2\pi\left(\frac{1}{3} - \frac{5}{12}\right) = -\frac{\pi}{6}.$$

5. This is a protractor contour, with an application of Jordan's Lemma. The integrand is $f(z) = \frac{ze^{iz}}{z^2+1}$, which has a simple pole at $z=i$ inside the protractor. It's easy to see that the residue is $\frac{ie^{i^2}}{2i} = \frac{1}{2e}$ and we have

$$\int_{C_1,R} \frac{ze^{iz}}{z^2+1}\,dz + \int_{C_2,R} \frac{ze^{iz}}{z^2+1}\,dz = \int_{C,R} \frac{ze^{iz}}{z^2+1}\,dz = 2\pi i\cdot\frac{1}{2e}.$$

The issue is whether the integral over the semicircle goes to 0 as $R \to \infty$. We have, by Jordan's Lemma,

$$\left| \int_{C_{2,R}} \frac{ze^{iz}}{z^2+1}\, dz \right| \leq \int_{C_{2,R}} \frac{R}{R^2-1} |e^{iz}|\, |dz| < \frac{\pi R}{R^2-1} \to 0.$$

(This is the book's version of the Jordan Lemma, not the one I gave in class; to make the translation, put $z = Re^{it}$, and be careful with the R's.) It follows that

$$\int_{-\infty}^{\infty} \frac{z\cos z + iz\sin z}{z^2+1}\, dz = i\frac{\pi}{e}.$$

By viewing the imaginary part above, we have

$$\int_{0}^{\infty} \frac{z\sin z}{z^2+1}\, dz = \frac{1}{2}\int_{-\infty}^{\infty} \frac{z\sin z}{z^2+1}\, dz = \frac{\pi}{2e}.$$

6. This can actually be done by Cauchy's Integral Formula, because the denominator has a pole at the center of the circle, and no others in and on the circle. With $f(z) = (z+i)^{-4}$, the correct answer is

$$2\pi i \frac{f'''(i)}{3!} = 2\pi i \frac{(-4)(-5)(-6)(2i)^{-7}}{3!} = \frac{5\pi}{16}.$$

7. The integral is Cauchy convergent if and only if the following limit exists as $R \to \infty$:

$$\int_{-R}^{R} f(x)\, dx = \int_{-R}^{R} f_e(x)\, dx + \int_{-R}^{R} f_o(x)\, dx = \int_{-R}^{R} f_e(x)\, dx + 0 = 2\int_{0}^{R} f_e(x)\, dx.$$

Thus, if the integral is Cauchy convergent, then $\lim_{R\to\infty} \int_0^R f_e(x)\, dx$ exists, and since f_e is even, this means that $\int_{-\infty}^{\infty} f_e(x)\, dx$ exists in the usual sense. Conversely, if this integral is convergent, then $\lim_{R\to\infty} \int_0^R f_e(x)\, dx$ exists and so the integral of f is Cauchy convergent.

8. I have given you a function $f(z) = \frac{z^2}{z^4+4}$ with four simple poles, at $\pm 1 \pm i$, for all four choices of sign. There are two poles inside the protractor, at $\pm 1 + i$, and as in the first problem, since f is a rational function whose denominator has degree 2 more than the numerator, we have

$$\int_{C_{1,R}} \frac{z^2}{z^4+4}\, dz + \int_{C_{2,R}} \frac{z^2}{z^4+4}\, dz = 2\pi i \left(Res(f(z), 1+i) + Res(f(z), -1+i) \right) \implies$$

$$\int_{-R}^{R} \frac{x^2}{x^4+4}\, dx + o(1) = 2\pi i \left(\frac{(1+i)^2}{4(1+i)^3} + \frac{(-1+i)^2}{4(-1+i)^3} \right) \implies$$

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+4}\, dx = \frac{\pi i}{2} \left(\frac{1}{1+i} + \frac{1}{-1+i} \right) = \frac{\pi i}{2} \cdot \frac{2i}{-2} = \frac{\pi}{2}.$$

9., 10. Note here that a and b have positive real parts so that ia and ib are in the upper half plane, and $-ia$ and $-ib$ are in the lower half plane, and not inside the protractor. We first assume that $a \neq b$, so that if we integrate the usual function over the usual contour, and compute the usual residues, we have

$$\int_{C_1,R} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\, dz + \int_{C_2,R} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\, dz =$$
$$= 2\pi i \cdot \left(\frac{e^{-a}/(-a^2+b^2)}{2ai} + \frac{e^{-b}/(-b^2+a^2)}{2bi} \right) = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right).$$

As usual, we take $R \to \infty$ and find that

$$\int_{\infty}^{\infty} \frac{\cos x + i \sin x}{(x^2+a^2)(x^2+b^2)}\, dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right).$$

Now something a little tricky happens. We *cannot* just take the real and imaginary parts above, because a and b are complex numbers and not real! However, $\frac{\sin x}{(x^2+a^2)(x^2+b^2)}$ is an odd function, so its integral from $-\infty$ to ∞ is 0, and we are left with the desired integral and the desired answer.

Now suppose $b \to a$. We have by L'Hôpital's rule,

$$\lim_{b \to a} \frac{\pi((be^b)^{-1} - (ae^a)^{-1})}{a^2-b^2} = \frac{\pi(-1(ae^a)^{-2}(e^a + ae^a))}{-2a} = \frac{\pi(a+1)}{a^3} e^{-a}.$$

If we do the same integral as above with $\frac{e^{iz}}{(z^2+a^2)^2}$, then we have a pole of order two at ia, and so the integral is equal to $2\pi i$ times the derivative of $e^{iz}(z+ai)^{-2}$, evaluated at $z = ai$. It's the same thing, as one would expect.

11. Following the hints from the book and the sheet, we wish to integrate $f(z) = \frac{e^{iz}}{z}$ over a protractor which has a bump going around 0. To be specific

$$0 = \int_C \frac{e^{iz}}{z}\, dz = \int_{C_1} \frac{e^{iz}}{z}\, dz + \int_{C_2} \frac{e^{iz}}{z}\, dz + \int_{C_3} \frac{e^{iz}}{z}\, dz + \int_{C_4} \frac{e^{iz}}{z}\, dz.$$

We parameterize as in class, so that on C_1, $z = -x$ as x goes from R to ϵ, C_2 is the semicircle $z = \epsilon e^{it}$ as t goes from π to 0, C_3 is $z = x$ as x goes from ϵ to R and C_4 is $z = Re^{it}$ as t goes from 0 to π. There is a pole at $z = 0$, which is outside the contour, so the total integral is 0. We have

$$\int_{C_1} \frac{e^{iz}}{z}\, dz + \int_{C_3} \frac{e^{iz}}{z}\, dz = \int_{\epsilon}^{R} \frac{e^{iz} - e^{-iz}}{z}\, dz = 2i \int_{\epsilon}^{R} \frac{\sin z}{z}\, dz.$$

By Lemma 4.1, because we are going clockwise, and by evaluating the residue

$$\lim_{\epsilon \to 0} \int_{C_2} \frac{e^{iz}}{z}\, dz = (-\pi)i(-i) = \pi$$

As in #5, we have

$$\left| \int_{C_4} \frac{e^{iz}}{z}\, dz \right| < \frac{\pi}{R}$$

So, as $\epsilon \to 0$ and $R \to \infty$, we get

$$2i \int_0^{\infty} \frac{\sin z}{z}\, dz = -\pi \implies \int_0^{\infty} \frac{\sin z}{z}\, dz = \frac{\pi}{2} \implies \int_{-\infty}^{\infty} \frac{\sin z}{z}\, dz = \pi,$$

12. I will take the same contour as in #11, with $f(z) = \frac{z^a}{z^2+1}$. There is a pole in the contour at $z = i$, and the residue is $\frac{i^a}{2i} = \frac{e^{\pi i a/2}}{2i}$. We also have that, if $z = re^{it}$, then $|z^a| = r^a$

$$\left| \int_{C_2} \frac{z^a}{z^2+1}\, dz \right| < \pi\epsilon \frac{\epsilon^a}{1-\epsilon^2} = \frac{\pi\epsilon^{1+a}}{1-\epsilon^2},$$

and since $a > -1$, this upper bound goes to 0 as $\epsilon \to 0$. Similarly,

$$\left| \int_{C_4} \frac{z^a}{z^2+1}\, dz \right| < \pi R \frac{R^a}{R^2-1} = \frac{\pi R^{1+a}}{R^2-1},$$

and, since $a < 1$, this upper bound goes to 0 as $r \to \infty$. This leaves the integrals along the real axis,

$$\int_{C_1} \frac{z^a}{a^2+1}\, dz + \int_{C_3} \frac{z^a}{a^2+1}\, dz = \int_R^{\epsilon} \frac{e^{i\pi a} x^a}{x^2+1}\, (-dx) + \int_{\epsilon}^{R} \frac{x^a}{x^2+1}\, dx$$
$$= (1 + e^{\pi a}) \int_{\epsilon}^{R} \frac{x^a}{x^2+1}\, dx.$$

If, at long last, we let $\epsilon \to 0$ and $R \to \infty$, we have

$$(1 + e^{\pi a}) \int_0^{\infty} \frac{x^a}{x^2+1}\, dx = 2\pi i \cdot \frac{e^{\pi i a/2}}{2i} = \pi e^{\pi i a/2} \implies$$
$$\int_0^{\infty} \frac{x^a}{x^2+1}\, dx = \pi \frac{e^{\pi i a/2}}{(1 + e^{\pi a})} = \frac{\pi}{e^{\pi a/2} + e^{-\pi a/2}} = \frac{\pi}{2\cos\frac{\pi a}{2}}.$$

Phew.