**Math 348** **Homework 1** **Due Monday, Jan. 31, 2000**

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The last four problems are the "graduate" problems and are intended to be harder than the others. The symbol $(\mathcal{E})$ means that at least part of this problem, up to possible numerical alterations, has appeared on one of my old Math 346 or Math 348 exams. All unexplained references are to pages in the book.

1. §1.1 – Problem 4.
2. $(\mathcal{E})$ Find explicit real numbers $r$ and $s$ with the property that

$$\frac{1}{2^{168}}\left(\frac{1+5i}{2+3i}\right)^{348} = r + is.$$

3. $(\mathcal{E})$ Write all complex numbers $z$ which satisfy the following equations in the form $z = re^{i\theta}$.

$$z^4 = -4; \qquad z^3 = 8i; \qquad z^{-2} = 2i.$$

4. §1.2 – Problem 3a (by "describe" I mean, "sketch".)
5. §1.3 – Problem 2 (first three parts.)
6. §1.4 – Problem 1 (only for $f(z) = \frac{1+z}{1-z}$.)
7. §1.4 – Problem 5 (only for the set $|Re(z)| < 1$, and only the first four transformations.)
8. $(\mathcal{E})$ Sketch the region $R$, consisting of the complex numbers $re^{i\theta}$ with $1 \le r \le 2$ and $\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$, and also sketch its range under the mapping $w = \frac{1}{z}$.

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9. Additional problem 1.4 (p.42). You may assume that the binomial coefficients $\binom{n}{k}$, defined by $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$, are symmetric; that is, $\binom{n}{k} = \binom{n}{n-k}$.

10. Since $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$, every polynomial in $x$ and $y$ can be expressed in terms of $z$ and $\bar{z}$. For example, $2x + 4y = (1-2i)z + (1+2i)\bar{z}$ and $x^2 + y^2 = z\bar{z}$. Express the following as polynomials in $z$ and $\bar{z}$:

$$x^2 - y^2, \qquad 2xy, \qquad x^3 - 3xy^2.$$

11.§1.3 – Problem 8. I think the fastest way to do this problem is not via Problem 7, but by noting that if you fix complex numbers $z_1, z_2, z_3$ with the property that $T = z_1 z_2 z_3$ is an equilateral triangle, and if the order "$z_1, z_2, z_3, z_1$" gives them in counterclockwise order, then there is a simple way to write $z_3 - z_2$ as a multiple of $z_2 - z_1$, which gives a linear equation of the form $az_1 + bz_2 + cz_3 = 0$ for appropriate numbers $(a, b, c)$. Do the same in case "$z_1, z_2, z_3, z_1$" gives them in clockwise order, and then multiply the linear equations together. It will help to write the cube roots of unity as

$$\omega = \frac{-1 + i\sqrt{3}}{2}, \qquad \omega^2 = \frac{-1 - i\sqrt{3}}{2},$$

and remember that $1 + \omega + \omega^2 = 0$.

12. Define integers $a_n$ and $b_n$ by $a_n + ib_n = (3+4i)^n$. Prove that, for all $n \ge 1$, $a_n \equiv 3$ $(mod\ 5)$ and $b_n \equiv 4$ $(mod\ 5)$. That is, there exist *integers* $c_n$ and $d_n$ so that $a_n = 5c_n + 3$ and $b_n = 5d_n + 4$. (Hint: $a_{n+1} + ib_{n+1} = (3+4i)(a_n + ib_n)$.)

Explain why this implies that $(3+4i)^n$ is never real, and hence why $\frac{\arctan 4/3}{\pi}$ is irrational.