Math 423, Bonus Handout of 10/20/04, Version 2.0, 10/25/04

Let M denote the surface $z = xy$. We will compute the shape operator and find its Gaussian and mean curvature.

We take the Monge patch $\vec{x}(u, v) = (u, v, uv)$, and note immediately that

$$\vec{x}_u(u, v) = (1, 0, v), \qquad \vec{x}_v(u, v) = (0, 1, u),$$

hence $\vec{x}_u \times \vec{x}_v = (-v, -u, 1)$. We take as our unit normal vector

$$\vec{U}(u, v) = \frac{\vec{x}_u \times \vec{x}_v}{||\vec{x}_u \times \vec{x}_v||} = \left(\frac{-v}{\sqrt{1 + u^2 + v^2}}, \frac{-u}{\sqrt{1 + u^2 + v^2}}, \frac{1}{\sqrt{1 + u^2 + v^2}} \right).$$

Some time ago, we defined directional derivatives for vector fields, (see *O'Neill*, pp. 150, 195, 196.) If $\vec{v} \in T_P(M)$ and f is a differentiable real function on M, then $\vec{v}[f]$ is the value of $\frac{d}{dt}(f\vec{\alpha})(0)$ for all curves $\vec{\alpha}$ which lie on M, so that $\vec{\alpha}(0) = P$ and $\vec{\alpha}'(0) = \vec{v}$.

If $\vec{Z} = (Z_1, Z_2, Z_3)$ is a vector field on $\mathbf{R}^3$, then $\vec{\nabla}_{\vec{v}}\vec{Z} = (\vec{v}[Z_1], \vec{v}[Z_2], \vec{v}[Z_3])$; namely, $(\vec{v} \cdot \vec{\nabla} Z_1, \vec{v} \cdot \vec{\nabla} Z_2, \vec{v} \cdot \vec{\nabla} Z_3.)$. This computational scheme was used in the first version of this handout with $\vec{v} = \vec{x}_u = (1, 0, v)$ and $\vec{x}_v = (0, 1, u)$. We expressed $\vec{U}$ at a point (a, b, c) and noted that it is the restriction to M of a vector field on $\mathbf{R}^3$ that depends only on (x, y) and not on z. (This is true for any Monge patch. The normal vector fields of $z = f(x, y) + c_1$ and $z = f(x, y) + c_2$ will always be parallel.) This makes the computation very easy and we saw that the only thing that mattered were $\frac{\partial \vec{U}}{\partial u}$ and $\frac{\partial \vec{U}}{\partial v}$.

But there is a more direct way to do this. Since M is given by a patch $\vec{x}(u, v)$ and $P = \vec{v}(u_0, v_0)$, if we take the u-**coordinate curve** $\vec{\alpha}(t) = \vec{v}(u_0 + t, v_0)$, then $\vec{\alpha}(0) = P$ and $\vec{\alpha}'(0) = \vec{x}_u(P)$. (And similarly for the v-coordinate curve.) The directional derivatives of $\vec{U}$ are directly computable by calculus:

$$\left.\frac{\partial \vec{U}}{\partial u}\right|_{(u,v)=(u_0,v_0)} = \left(\frac{u_0 v_0}{(1 + u_0^2 + v_0^2)^{3/2}}, \frac{-(1 + v_0^2)}{(1 + u_0^2 + v_0^2)^{3/2}}, \frac{-u_0}{(1 + u_0^2 + v_0^2)^{3/2}} \right)$$

and

$$\left.\frac{\partial \vec{U}}{\partial v}\right|_{(u,v)=(u_0,v_0)} = \left(\frac{-(1 + u_0^2)}{(1 + u_0^2 + v_0^2)^{3/2}}, \frac{u_0 v_0}{(1 + u_0^2 + v_0^2)^{3/2}}, \frac{-v_0}{(1 + u_0^2 + v_0^2)^{3/2}} \right).$$

We now drop the subscripts for simplicity. These two vectors have to lie in $T_P(M)$; in fact,

$$\frac{\partial \vec{U}}{\partial u} = \left(\frac{uv}{(1 + u^2 + v^2)^{3/2}} \right) \vec{x}_u + \left(\frac{-(1 + v^2)}{(1 + u^2 + v^2)^{3/2}} \right) \vec{x}_v.$$

(Note that $uv(v) + (-(1 + v^2))u = -u$, so the third component is correct.) Similarly,

$$\frac{\partial \vec{U}}{\partial v} = \left(\frac{-(1 + u^2)}{(1 + u^2 + v^2)^{3/2}} \right) \vec{x}_u + \left(\frac{uv}{(1 + u^2 + v^2)^{3/2}} \right) \vec{x}_v.$$

Since
$$S_P(\vec{x}_u) = -\vec{\nabla}_{\vec{u}}(\vec{U}) = -\frac{\partial \vec{U}}{\partial u}, \qquad S_P(\vec{x}_v) = -\vec{\nabla}_{\vec{v}}(\vec{U}) = -\frac{\partial \vec{U}}{\partial v},$$
we may write the matrix for the operator S_P at (u, v):
$$\begin{pmatrix} \dfrac{-uv}{(1+u^2+v^2)^{3/2}} & \dfrac{(1+u^2)}{(1+u^2+v^2)^{3/2}} \\ \dfrac{(1+v^2)}{(1+u^2+v^2)^{3/2}} & \dfrac{-uv}{(1+u^2+v^2)^{3/2}} \end{pmatrix}.$$

As usual, H is one half the trace (sum of the diagonal elements) and K is the determinant. Thus.
$$H = \frac{1}{2} \cdot \frac{-2uv}{(1+u^2+v^2)^{3/2}} = \frac{-uv}{(1+u^2+v^2)^{3/2}}$$
and
$$\begin{aligned} K &= \frac{-uv}{(1+u^2+v^2)^{3/2}} \cdot \frac{-uv}{(1+u^2+v^2)^{3/2}} - \frac{(1+v^2)}{(1+u^2+v^2)^{3/2}} \cdot \frac{(1+u^2)}{(1+u^2+v^2)^{3/2}} \\ &= \frac{u^2v^2 - (1+u^2)(1+v^2)}{(1+u^2+v^2)^3} = \frac{u^2v^2 - 1 - u^2 - v^2 - u^2v^2}{(1+u^2+v^2)^3} \\ &= \frac{-1-u^2-v^2}{(1+u^2+v^2)^3} = \frac{-1}{(1+u^2+v^2)^2}. \end{aligned}$$

The principal curvatures, k_1 and k_2, are the roots of the quadratic $X^2 - 2HX + K = 0$, and it is not hard to see that these work out to
$$\frac{uv \pm \sqrt{(1+u^2)(1+v^2)}}{(1+u^2+v^2)^{3/2}}.$$

It's relevant that K is always < 0, and as (u, v) goes to infinity, $K \to 0$. It's also relevant that the mean curvature is 0 only when $uv = 0$; that is, on the u- and v- axes, which map to the x- and y- axes.

What here is special about this surface? Not much. In fact, this computation would work for any Monge patch $(u, v, f(u, v))$ for the surface $z = f(x, y)$ defined for $(u, v) \in \mathbf{R}^2$. (If $D \neq \mathbf{R}^2$, we have to restrict the domain of our assertions accordingly.) The only thing special is that f_u, f_v and $\sqrt{1 + f_u^2 + f_v^2}$ are not especially unpleasant to calculate. So, to answer your questions:

(1) Yes, you can do the same thing here on the homework for $f(u, v) = u^2 + v^2$. You do **not** have to repeat my explanations for the computations.

(2) If you want to work out what happens in general, remember that the computational techniques in 5.4 allow one to find H and K more quickly for any Monge patch. (However, they do not give the principal directions – which are found from the eigenvectors of the shape operator matrix – without further study; this can be done however, and we'll do it in class today.)