Math 423, Bonus Handout of 10/20/04, Version 3.0, 11/8/04

Let M denote the surface $z = xy$. We will compute the shape operator and find its Gaussian and mean curvature.

We take the Monge patch $\vec{x}(u, v) = (u, v, uv)$, and note immediately that

$$\vec{x}_u(u, v) = (1, 0, v), \qquad \vec{x}_v(u, v) = (0, 1, u),$$

hence $\vec{x}_u \times \vec{x}_v = (-v, -u, 1)$. We take as our unit normal vector

$$\vec{U}(u, v) = \frac{\vec{x}_u \times \vec{x}_v}{||\vec{x}_u \times \vec{x}_v||} = \left(\frac{-v}{\sqrt{1 + u^2 + v^2}}, \frac{-u}{\sqrt{1 + u^2 + v^2}}, \frac{1}{\sqrt{1 + u^2 + v^2}} \right).$$

Some time ago, we defined directional derivatives for vector fields, (see *O'Neill*, pp. 150, 195, 196.) If $\vec{v} \in T_P(M)$ and f is a differentiable real function on M, then $\vec{v}[f]$ is the value of $\frac{d}{dt}(f\vec{\alpha})(0)$ for all curves $\vec{\alpha}$ which lie on M, so that $\vec{\alpha}(0) = P$ and $\vec{\alpha}'(0) = \vec{v}$.

If $\vec{Z} = (Z_1, Z_2, Z_3)$ is a vector field on $\mathbf{R}^3$, then $\vec{\nabla}_{\vec{v}}\vec{Z} = (\vec{v}[Z_1], \vec{v}[Z_2], \vec{v}[Z_3])$; namely, $(\vec{v} \cdot \vec{\nabla} Z_1, \vec{v} \cdot \vec{\nabla} Z_2, \vec{v} \cdot \vec{\nabla} Z_3.)$. This computational scheme was used in the first version of this handout with $\vec{v} = \vec{x}_u = (1, 0, v)$ and $\vec{x}_v = (0, 1, u)$. We expressed $\vec{U}$ at a point (a, b, c) and noted that it is the restriction to M of a vector field on $\mathbf{R}^3$ that depends only on (x, y) and not on z. (This is true for any Monge patch. The normal vector fields of $z = f(x, y) + c_1$ and $z = f(x, y) + c_2$ will always be parallel.) This makes the computation very easy and we saw that the only thing that mattered were $\frac{\partial \vec{U}}{\partial u}$ and $\frac{\partial \vec{U}}{\partial v}$.

But there is a more direct way to do this. Since M is given by a patch $\vec{x}(u, v)$ and $P = \vec{v}(u_0, v_0)$, if we take the u-**coordinate curve** $\vec{\alpha}(t) = \vec{v}(u_0 + t, v_0)$, then $\vec{\alpha}(0) = P$ and $\vec{\alpha}'(0) = \vec{x}_u(P)$. (And similarly for the v-coordinate curve.) The directional derivatives of $\vec{U}$ are directly computable by calculus:

$$\left.\frac{\partial \vec{U}}{\partial u}\right|_{(u,v)=(u_0,v_0)} = \left(\frac{u_0 v_0}{(1 + u_0^2 + v_0^2)^{3/2}}, \frac{-(1 + v_0^2)}{(1 + u_0^2 + v_0^2)^{3/2}}, \frac{-u_0}{(1 + u_0^2 + v_0^2)^{3/2}} \right)$$

and

$$\left.\frac{\partial \vec{U}}{\partial v}\right|_{(u,v)=(u_0,v_0)} = \left(\frac{-(1 + u_0^2)}{(1 + u_0^2 + v_0^2)^{3/2}}, \frac{u_0 v_0}{(1 + u_0^2 + v_0^2)^{3/2}}, \frac{-v_0}{(1 + u_0^2 + v_0^2)^{3/2}} \right).$$

We now drop the subscripts for simplicity. These two vectors have to lie in $T_P(M)$; in fact,

$$\frac{\partial \vec{U}}{\partial u} = \left(\frac{uv}{(1 + u^2 + v^2)^{3/2}} \right) \vec{x}_u + \left(\frac{-(1 + v^2)}{(1 + u^2 + v^2)^{3/2}} \right) \vec{x}_v.$$

(Note that $uv(v) + (-(1 + v^2))u = -u$, so the third component is correct.) Similarly,

$$\frac{\partial \vec{U}}{\partial v} = \left(\frac{-(1 + u^2)}{(1 + u^2 + v^2)^{3/2}} \right) \vec{x}_u + \left(\frac{uv}{(1 + u^2 + v^2)^{3/2}} \right) \vec{x}_v.$$

Since

$$S_P(\vec{x}_u) = -\vec{\nabla}_{\vec{u}}(\vec{U}) = -\frac{\partial \vec{U}}{\partial u}, \qquad S_P(\vec{x}_v) = -\vec{\nabla}_{\vec{v}}(\vec{U}) = -\frac{\partial \vec{U}}{\partial v},$$

we may write the matrix for the operator S_P at (u, v):

$$\begin{pmatrix} \frac{-uv}{(1+u^2+v^2)^{3/2}} & \frac{(1+u^2)}{(1+u^2+v^2)^{3/2}} \\ \frac{(1+v^2)}{(1+u^2+v^2)^{3/2}} & \frac{-uv}{(1+u^2+v^2)^{3/2}} \end{pmatrix}.$$

As usual, H is one half the trace (sum of the diagonal elements) and K is the determinant. Thus.

$$H = \frac{1}{2} \cdot \frac{-2uv}{(1+u^2+v^2)^{3/2}} = \frac{-uv}{(1+u^2+v^2)^{3/2}}$$

and

$$\begin{aligned} K &= \frac{-uv}{(1+u^2+v^2)^{3/2}} \cdot \frac{-uv}{(1+u^2+v^2)^{3/2}} - \frac{(1+v^2)}{(1+u^2+v^2)^{3/2}} \cdot \frac{(1+u^2)}{(1+u^2+v^2)^{3/2}} \\ &= \frac{u^2v^2 - (1+u^2)(1+v^2)}{(1+u^2+v^2)^3} = \frac{u^2v^2 - 1 - u^2 - v^2 - u^2v^2}{(1+u^2+v^2)^3} \\ &= \frac{-1-u^2-v^2}{(1+u^2+v^2)^3} = \frac{-1}{(1+u^2+v^2)^2}. \end{aligned}$$

The principal curvatures, k_1 and k_2, are the roots of the quadratic $X^2 - 2HX + K = 0$, and it is not hard to see that these work out to

$$\frac{-uv \pm \sqrt{(1+u^2)(1+v^2)}}{(1+u^2+v^2)^{3/2}}.$$

It's relevant that K is always < 0, and as (u, v) goes to infinity, $K \to 0$. It's also relevant that the mean curvature is 0 only when $uv = 0$; that is, on the u- and v- axes, which map to the x- and y- axes.

What here is special about this surface? Not much. In fact, this computation would work for any Monge patch $(u, v, f(u, v))$ for the surface $z = f(x, y)$ defined for $(u, v) \in \mathbf{R}^2$. (If $D \neq \mathbf{R}^2$, we have to restrict the domain of our assertions accordingly.) The only thing special is that f_u, f_v and $\sqrt{1 + f_u^2 + f_v^2}$ are not especially unpleasant to calculate.

Note: final paragraphs of Version 2.0 deleted and sign corrected on the values of the principal curvatures. New material begins here.

Observe from the above that for any (a, b), we have

$$S_p(a\vec{x}_u + b\vec{x}_v) = \left(\frac{-auv + b(1+u^2)}{(1+u^2+v^2)^{3/2}}\right)\vec{x}_u + \left(\frac{a(1+v^2) - buv}{(1+u^2+v^2)^{3/2}}\right)\vec{x}_v,$$

so if $S_p(a\vec{x}_u + b\vec{x}_v) = \lambda(a\vec{x}_u + b\vec{x}_v)$, then we have two equations:

$$\lambda a = \frac{-auv + b(1+u^2)}{(1+u^2+v^2)^{3/2}}; \qquad \lambda b = \frac{a(1+v^2) - buv}{(1+u^2+v^2)^{3/2}}.$$

This quickly simplifies to

$$b(-auv + b(1+u^2)) = a(a(1+v^2) - buv),$$

so $b^2(1+u^2) = a^2(1+v^2)$, and we may take as two independent solutions $a = \sqrt{1+u^2}$ and $b = \pm\sqrt{1+v^2}$, yielding as principal curvatures

$$\frac{-uv \pm \sqrt{(1+u^2)(1+v^2)}}{(1+u^2+v^2)^{3/2}}.$$

Please note that the average of these two is H above, and their product is

$$\frac{u^2v^2 - (1+u^2)(1+v^2)}{(1+u^2+v^2)^3} = -\frac{1}{(1+u^2+v^2)^2}.$$

Furthermore, these principal vectors are, explicitly

$$(\sqrt{1+u^2}, \pm\sqrt{1+v^2}, v\sqrt{1+u^2} \pm u\sqrt{1+v^2}),$$

which are orthogonal and also appear, in garbled form, on p.224, #10. As a vector field on the surface $z = xy$, you can replace u with x and v with y above, since it's a Monge patch.

Now what are the principal curves? These are 3-dimensional and we first look at the projection onto the first two components. We see that at any point (x_0, y_0), the first two components of the velocity vector are multiples of $(\sqrt{1+x^2}, \pm\sqrt{1+y^2})$, and hence the slope of the curve at (x_0, y_0) is $\pm\frac{\sqrt{1+y^2}}{\sqrt{1+x^2}}$. In other words,

$$\frac{dy}{dx} = \pm\frac{\sqrt{1+y^2}}{\sqrt{1+x^2}}.$$

We can solve this differential equation. Note that under the substitution $t = \sinh w$,

$$\int \frac{dt}{\sqrt{1+t^2}} = \int \frac{\cosh w \; dw}{\cosh w} = w + c = \sinh^{-1}(t) + c.$$

We have a separable differential equation

$$\frac{dy}{\sqrt{1+y^2}} = \pm\frac{dx}{\sqrt{1+x^2}} \implies \sinh^{-1}(y) = \pm\sinh^{-1} x + c \implies y = \pm\sinh(\sinh^{-1} x + c).$$

Using the addition formula for sinh, which is the same as that for sin, formally, one obtains, with $b = \sinh c$,

$$y = \pm(b\sqrt{x^2+1} + \left(\sqrt{b^2+1}\right) x)$$