Math 448 **Homework 10** **Due Fri., Nov. 9, 2007**

(ungraded) §3.1 – 1, 13

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1. (graded) §3.1 –2. It is easier to do this by Rouché's theorem. Let the "main" function be the first and last terms, and the "perturbation" be the middle.

2. (graded) §3.1 – 20.

3. (graded) ($\mathcal{E}$) Suppose f is entire and $|f(z)| \leq 1$ on the unit circle $|z| = 1$. Using Rouché's Theorem, prove that there is exactly one z_0 with $|z_0| < 1$ and

$$f(z_0) = \frac{10z_0}{2 + z_0}.$$

Hint: It will help to define $\phi(z) = \frac{10z}{z+2}$.

4. (graded) ($\mathcal{E}$) Let $f(z) = \frac{1}{z} + \frac{1}{z-1}$. Find, carefully, Laurent series for f which converge in each of the following regions: (a) $|z - 2| < 1$, (b) $1 < |z - 2| < 2$, (c) $2 < |z - 2|$.

5. (graded) ($\mathcal{E}$) Let R denote the (closed, solid) square in the complex plane with vertices $1 \pm i$ and $-1 \pm i$.

a. Determine the maximum and minimum of $|e^z|$ on R.
b. Determine the maximum and minimum of $|z^2|$ on R.
c. Use Rouché's Theorem to determine the number of zeros of $f(z) = e^z + 100z^2$ in R.

6. (graded) ($\mathcal{E}$) Determine the linear fractional transformation $T(z) = \frac{az+b}{cz+d}$ so that $T(0) = 1$, $T(1) = i$ and $T(\infty) = -1$, and determine $T(i)$.

7. (graded) What is the name of the author of the textbook. This is not a trick question, but a way of thinning the homework.

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8. (bonus) Suppose f is an entire function, $f(0) = f'(0) = f''(0) = 0$ and $f'''(0) = 1$. Show that there exists z_0 with $|z_0| = 2$ and so that $|f(z_0)| \geq \frac{4}{3}$. Hint: consider $g(z) = z^{-3}f(z)$ (for $z \neq 0$), and examine the singularities of g at 0.

9. (bonus) Find positive integers $0 < r_1 < r_2$ with the property that the polynomial $f(z) = z^5 + 2z^3 + 4z + 100$ has no zeros in the region $|z| \leq r_1$ and five zeros in the region $|z| < r_2$. Needless to say, this problem has more than one correct answer.

10. (bonus) Work the integral of §2.6 27 thorugh 31 with $f(z) = \frac{1}{(3z-1)^2}$ and simplify your final answer to get information about the sum of a subset of $\{\frac{1}{n^2}\}$.