Math 448 **Homework 4** **Due Fri., Sept. 21, 2007**

Same instructions as last time. For infinite series problems, you may always use formulas that have already been established.

(ungraded) §2.1 – 17, §2.2 – 3, 5, 7, 9, 17

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1. (graded) §2.1 – 20 ad.

2. (graded) §2.2 – Problems 2 and 4. (Closed form not requested.)

3. (graded) §2.2 – Problems 8 and 10.

4. (graded) §2.2 – Problems 14 and 16.

5. (graded) ($\mathcal{E}$) Let C denote that portion of the circle $|z| = 2$ running from $z = 2$ to $z = 2i$ in the usual counterclockwise fashion. Compute, from the definition,

$$\int_C (\bar{z} + z^2)\, dz.$$

6. (graded) ($\mathcal{E}$) Verify that $u(x, y) = xy + 7x + e^{-y} \sin x$ is a harmonic function, find all harmonic conjugates $v(x, y)$ and find an expression for $f(z) = f(x+iy) = u(x, y)+iv(x, y)$ which depends only on z and certain real constants.

7. (graded) ($\mathcal{E}$) Suppose $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is an entire function with the property that $v(x, y) = 2(u(x, y))^2$ for all (x, y). Use the Cauchy-Riemann equations to prove that f must be constant.

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8. (bonus) Determine all *entire* functions $f(z) = f(x, y) = u(x, y) + iv(x, y)$ with the property that $(x^2 - y^2)u(x, y) - 2xyv(x, y) = 0$ for all (x, y). Hints: what can you say about $z^2 f(z)$? What happens at $z = 0$?

9. (bonus) Determine all real homogeneous harmonic polynomials of degree 7; that is,

$$\left\{ f(x, y) = \sum_{j=0}^{7} a_j x^j y^{7-j} \ : \ \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \right\}.$$

(Hint: it will be a vector space over $\mathbf{R}$ of dimension two.)

10. (bonus) a. Verify *from the definition* the identity

$$\sin 3z = 3 \sin z - 4 \sin^3 z.$$

b. Use this identity and the formula at the bottom of p.100 (or any other correct and explained method) to give a closed form for the power series for $\sin^3(z)$ at $z = 0$.

c. Determine, by any correct method, a numerical expression for $f^{(2007)}(0)$, where $f(z) = \sin^3 z$. (Something *like* "$\frac{7^{23} \cdot e^{34}}{37!}$" is what I mean by a numerical expression.)