Math 448 **Homework 9** **Due Fri., Nov. 2, 2007**

(ungraded) §2.6 – 7, 13

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1. (graded) §2.6 – 14.

2. (graded) §2.6 – 22ab. [The point is to make a suitable substitution; no new integrals are needed!]

3. (graded) ($\mathcal{E}$) Evaluate, with explanation, the following definite integral:

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)(x^2+9)}.$$

4. (graded) ($\mathcal{E}$) Let C be any simple closed curve in the complex plane, taken in the positive orientation. We define a function $f(z)$ for all z not on the curve C as follows:

$$f(z) = \frac{1}{2\pi i} \int_C \frac{e^{2\zeta}}{(\zeta - z)^3}\, d\zeta.$$

Compute $f(z)$. (Your answer will depend on whether z is inside C or outside C.)

5. (graded) ($\mathcal{E}$) A mindless calculation gives

$$1 + \frac{1}{z^2} + \frac{1}{z^4} + \cdots = \frac{1}{1 - \frac{1}{z^2}} = \frac{z^2}{z^2 - 1}.$$

Does this mean that $f(z) = \frac{z^2}{z^2-1}$ has an essential singularity at $z = 0$? Explain your answer carefully, with a discussion of the relevance both of the computation and of the singularity.

6. (graded) ($\mathcal{E}$) A function f has the Laurent series

$$f(z) = \sum_{n=0}^{\infty} \frac{(z-1)^n}{3^n} + 2\sum_{n=1}^{\infty} \frac{2^n}{(z-1)^n}.$$

Determine the set of z for which this series converges, and sum the series in closed form.

7. (graded) ($\mathcal{E}$) Let C_r denote the semicircle of radius r, centered at the origin and going counterclockwise from $z = r$ to $z = -r$ in the upper half plane. Find (with explanation) real numbers M_1 and M_2 so that

$$\left| \int_{C_{50}} \frac{e^{iz}\mathrm{Log}(z)\, dz}{(z^2+7)^3} \right| \le M_1$$

and

$$\left| \int_{C_{.01}} \frac{e^{iz}\mathrm{Log}(z)\, dz}{(z^2+7)^3} \right| \le M_2.$$

Hints: you could think of the contours as C_R and C_ϵ respectively, with $R = 50$ and $\epsilon = .01$. Don't worry about antiderivatives.

8. (bonus) Evaluate

$$\int_0^\infty \frac{x \sin 3x \ dx}{x^2 + 16}.$$

9.& 10. (bonus) Let C_R $(R > \sqrt{2})$ be the half-protractor contour sketched below, with pieces $C_{1,R}$, $C_{2,R}$ and $C_{3,R}$ identified.

a. Show that

$$f(z) = \frac{z}{z^4 + 4}$$

has exactly one singularity within C_R and compute its residue there. (Hint: it has the form $m + ni$, where m and n are integers.)

b. Find a complex number λ so that

$$\int_{C_{3,R}} f(z) \ dz = \lambda \int_{C_{1,R}} f(z) \ dz$$

(This will involve carefully parameterizing both line segments.)

c. Find a function $\Phi(R)$ with the properties that

$$\left| \int_{C_{2,R}} f(z) \ dz \right| < \Phi(R) \qquad \text{and} \qquad \lim_{R \to \infty} \Phi(R) = 0.$$

d. Put everything together to evaluate

$$\int_0^\infty \frac{x}{4 + x^4} \ dx.$$