Math 448
HW 7
Comments

Much to say.

#1. If $f(z) = \frac{P(z)}{Q(z)}$ and P and Q are entire then the singularities of f are isolated and occur at all points where $Q(z_0) = 0$.

The nature of the singularity depends on the order of the zero of Q at z_0 and the order (if any) of the zero at z_0.

(a) $P(z_0) \neq 0$ $Q(z_0)$ zero order m

$\Rightarrow f(z)$ has pole order m at z_0

(b) $P(z_0) = 0$ P zero order l

then $f(z) = \frac{(z-z_0)^l(a_l + a_{l+1}(z-z_0)+\cdots)}{(z-z_0)^m(b_m + b_{m+1}(z-z_0)+\cdots)}$

$a_l, b_l \neq 0$

so $l < m$ pole order $m-l$

$l = m$ removable sing $\neq 0$

$l > m$ removable sing (zero order $l-m$).

#2 This applies here.

$Q_1(z)$ has zeros of order 1 $\sin z$ at $n\pi$ for all $n \in \mathbb{Z}$.

$P(z) = z^2$ has zero of order 2 at $0 \cdot \pi$, no zero at $n\pi$, $n \in \mathbb{Z}$, $n \neq 0$.

$P(z) = e^z - 1$, $Q(z) = e^{2z} - 1 = (e^z+1)(e^z-1)$

Q has zeros whenever $e^z = -1$ or $e^z = 1$.

If $e^z = 1$, $P(z) = 0$; if $e^z = -1$ $P(z) \neq 0$

Examination of P', Q' there shows that all zeros have order 1. The result follows.

#3 The choice of $|z| > 2$ was technically required, even if the formulas were more generally valid

#4. You worked too hard to find a function with a pole at $z_0 = 2$ of order 3 + residue 7. You can just write down $\frac{1}{(z-2)^3} + \frac{7}{z-2}$!

#5 You have to draw pictures to really see where the poles are and what's inside C

#6 Done in class I think.

#7 Slightly botched. $|w| > 5/2$

$\frac{1}{1-2z} + \frac{1}{2+z}$ at $z_0 = 2$, let $w = z+2$
$z = w-2$

$\frac{1}{2+w} = \frac{1}{w}$ $\frac{1}{1-2z} = \frac{1}{1-2(w-2)} = \frac{1}{5-2w}$ big w, so

$\frac{1}{5-2w} = -\frac{1}{2w}\frac{1}{1-\frac{5}{2w}} = -\frac{1}{2w}\sum_{n=0}^{\infty}\left(\frac{5}{2w}\right)^n$

$= -\sum_{n=0}^{\infty}\frac{5^n}{2^{n+1}}\cdot\frac{1}{w^{n+1}}$ $m = n+1$ $= -\sum_{m=1}^{\infty}\frac{5^{m-1}}{2^m}\cdot\frac{1}{w^m}$

combining with $\frac{1}{w}$

$\left(1-\frac{1}{2}\right)\frac{1}{w} - \sum_{m=2}^{\infty}\frac{5^{m-1}}{2^m}\cdot\frac{1}{w^m}$.

#8 b, c I got the sign wrong

b) $A(z) = -e^z \Rightarrow f(z) = C \cdot e^{-e^z}$

c) $A(z) = -\frac{1}{z} \Rightarrow f(z) = C e^{-(\log z - \log z_0)}$

Note: $\log(e^w)$ need not equal w, but $e^{\log w}$ always equals w.

#9 See p. 70 (bottom) for a really easy way to do this.

#10 This is a special case of a more general result, as we'll see.

Math 448
HW 8
Bonus

Two main points

(1) You must justify in any integral of those kind that there is a portion of the integral which goes to 0 as $R \to \infty$ and/or $\varepsilon \to 0$, etc.. We have specific Theorems you can quote for polynomial $P(z), Q(z)$

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx \quad (\deg Q \geq \deg P + 2) \qquad \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{ix}\,dx \quad (\deg Q \geq \deg P + 1)$$

(2) If $f(x) \geq 0$, then $\int_0^b f(x)\,dx \geq 0$ and must be real. If your calculation says otherwise, you know you've made a mistake.

§26-2 Horrible calculation But be careful... $\sqrt{2+i}$ is not a well-defined object even if Mathematica thinks so.

§26-4. Mr. Huang's great idea:

$$\frac{\cos \alpha x}{(x^2+1)(x^2+4)} = \frac{1}{3}\left(\frac{\cos \alpha x}{x^2+1} - \frac{\cos \alpha x}{x^2+4}\right)$$

both of which are integrable and done in example 3

§26-10 Awful calculation but nice answer

#4, 5 Most of you got; always be alert to the direction of the contour.

#6, 7. See comments (1) and (2) above.

#8 Please keep in mind that if f is not analytic at z_0 and f is bounded near z_0 it does not follow that f has a removable singularity at z_0 unless z_0 is isolated. Think $\operatorname{Log} z$ at $z_0 = -1$.

#9 See (1), (2).

#10 I did this in class and only 4 in the class submitted it as a solution. Read those problems!

Math 448
HW 9
More

1. It is always necessary to justify the use of the contour method by explaining why the integrals we don't care about go to 0 as $R\to\infty$, $\varepsilon\to 0$, $\delta\to 0$, etc.

2. The integral of a positive real function will be positive on a real integral oriented correctly.

A non-homework problem. (What's wrong here?)

$$\int_{-\infty}^{\infty}\frac{x\,dx}{x^2+1} = 2\pi i\cdot \operatorname{Res}\left(\frac{z}{z^2+1}; i\right) = 2\pi i\cdot\frac{i}{2i}\begin{matrix}\leftarrow z|_i\\ \leftarrow (z^2+1)'|_i\end{matrix}$$
$$= \pi i$$

(a) The residue is correctly evaluated

(b) The integral of a real function seems to be imaginary.

(c) $\deg(x^2+1)=2$ and $\deg(x)=1$ and 2 is not $\geq 1+2$. However, just because a condition is not applicable, it doesn't follow that it's unusable.

(d) In fact  $\displaystyle\int_{-R}^{R}\frac{x\,dx}{x^2+1}+\int_{\frown}\frac{z\,dz}{z^2+1}=\pi i$

and since $\frac{x}{x^2+1}$ is odd, $\displaystyle\int_{-R}^{R}\frac{x\,dx}{x^2+1}=0$, so $\displaystyle\int_{\theta=0}^{\pi}\frac{Re^{i\theta}\, iRe^{i\theta}}{R^2e^{2i\theta}+1}\,d\theta=\pi i$

If you like this sort of thing, write in its real and imaginary parts. I'd divide numerator and denominator by $e^{i\theta}$.

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3. I feel obliged to list common errors: two people wrote $\sin\frac{\pi}{2}=\frac{\sqrt{2}}{2}$.

4. #3 can be done by parts. $\frac{1}{(x^2+1)(x^2+4)(x^2+9)}=\frac{c_1}{x^2+1}+\frac{c_2}{x^2+4}+\frac{c_3}{x^2+9}$.

5. #4 shows that one can give a <u>formula</u> for an analytic function whose behavior inside a curve and out are really different.

6. I goofed on #6, should be $\sum_{n=1}^{\infty}\frac{2^n}{(z-1)^n}$ (not $n=0$) to sum to $\frac{2}{z-3}$, so we get $\frac{3}{4-z}+\frac{4}{z-3}$

7. Be careful in the derivation of the estimates, though it doesn't affect the conclusion

Math 348
HW 10
2nd Chance

1. Okay, there were two botched solutions on my part, so I'd like to do these first.

#6. $T(z)=\frac{az+b}{cz+d}$

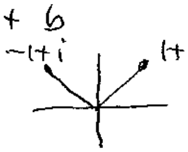
$T(0)=1$, $T(\infty)=-1$ implies $c=-a$, $d=a$ as before.

$$T(z)=\frac{az+b}{-az+b}$$

$T(1)=i \Rightarrow i=\frac{a+b}{-a+b}$

$-ia+ib=a+b$

$(-1+i)\,b=(1+i)\,a.$



$\Rightarrow b=-ia$, not $b=ia$!

$$b=\frac{1+i}{-1+i}a=\frac{\sqrt{2}\,e^{i\frac{\pi}{4}}}{\sqrt{2}\,e^{i\frac{3\pi}{4}}}=e^{-i\frac{\pi}{2}}=-i$$

So $T(z)=\frac{az-ia}{-az-ia}=-\frac{z-i}{z+i}$

(This is $\frac{1}{T(z)}$ from the solution given.)

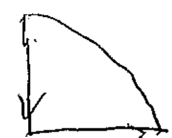
It follows that $T(i)=-\frac{i-i}{i+i}=0$ (not ∞).

However, you should all be able to reduce $\frac{1+i}{1-i}$, etc, if asked. I'll take points off myself if necessary.

#3. I inaccurately looked for solutions to $\frac{10z}{2+z}=z$.
The point should be that,
$\phi(z)$ is analytic for $z\neq-2$ (which doesn't affect the set $|z|\le 1$)
and $|\phi(z)|\geq\frac{10}{3}$ on $|z|=1$
$|f(z)|\leq 1$ on $|z|=1$

so the number of zeros of $\phi(z)$ equals the number of zeros of $(\phi-f)(z)$ inside $|z|=1$.
And $\phi(z)=0 \iff z=0$, so this number is 1.

#1. By argument method
$\phi(z)=z^4-3z^2+3$ is positive real for $z\in\mathbb{R}, i\mathbb{R}$ by the argument of the solutions given last time.



C_1 so on C_1, $\arg(\phi(z))=0$.

For $z=Re^{i\alpha}$

$$\phi(z)=R^4 e^{i4\alpha}\underbrace{\left(1-\frac{3}{R^2e^{2i\alpha}}+\frac{3}{R^4e^{4i\alpha}}\right)}_{1+\epsilon}$$

where ϵ is small, so
$\arg\phi(Re^{i\alpha})\approx 4\alpha$
and as α increases from 0 to $\frac{\pi}{2}$, this increases from 0 to 2π

on C_3, $\arg\phi(z)=2\pi$, so net increase of argument is $2\pi \Rightarrow 1$ zero.

#4 I slightly prefer $(-1)^n(z-2)^n$ to $(2-z)^n$ and $\sum_{n=1}^{\infty}a_nz^{-n}$ or $\sum_{n=1}^{\infty}\frac{a_n}{z^n}$ to $\sum_{n=-\infty}^{-1}a_nz^n$, because you can't "start" at $-\infty$.

#5 roots are at $z=-.00497\pm.09963\,i$ where $z^2\neq-\frac{1}{100}$ and $e^z\neq 1$.

#10, 8 I'll talk about in class.

Some more definite integrals

Math 448
10/24/07

1. $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^m}$. We first need a lemma:
$m \geq 1$ If $f(x) = (ax+b)^{-m}$, then

$$f^{(k)}(x) = (-a)^k \cdot \frac{(m+k-1)!}{(m-1)!}(ax+b)^{-(m+k)}$$

This is true in the base case ($k=0$, or $k=1$ if you prefer); if true for k, then

$$f^{(k+1)}(x) = (-a)^k \frac{(m+k-1)!}{(m-1)!}(-1)(m+k)(a)(ax+b)^{-(m+k+1)} \quad ✓.$$

Now $\deg 1 = 0$, $\deg (1+x^2)^m = 2m \geq 2$, so by our result,
$(1+z^2)^{-m}$ has poles at $i, -i$

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^m} = 2\pi i \cdot \operatorname{Res}\left(\frac{1}{(1+z^2)^m}; i\right) \quad \text{we ignore } -i.$$

$$\frac{1}{(1+z^2)^m} = \frac{\frac{1}{(z+i)^m}}{(z-i)^m} \qquad \operatorname{Res}\left(\frac{1}{(1+z^2)^m}; i\right) = \frac{H^{(m-1)}(i)}{(m-1)!} \qquad H(z) = \frac{1}{(z+i)^m}$$

By the lemma, $H^{(m-1)}(z) = (-1)^{m-1} \frac{(2m-2)!}{(m-1)!}(z+i)^{-(2m-1)}$. Thus

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^m} = 2\pi i \cdot \frac{1}{(m-1)!} \cdot (-1)^{m-1} \cdot \frac{(2m-2)!}{(m-1)!} \cdot \frac{1}{(2i)^{2m-1}}$$

$$= \pi \cdot \frac{(2m-2)!}{((m-1)!)^2} \cdot \frac{2 \cdot i \cdot i^{2m-2}}{2^{2m-1} i^{2m-1}} = \frac{\pi}{2^{2m-2}} \binom{2m-2}{m-1} \qquad (-1) = i^2$$

An alternate derivation. Let $x = \tan\theta$, $-\infty < x < \infty \Leftrightarrow -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
$dx = \sec^2\theta \quad x^2+1 = \sec^2\theta$,

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^m} = \int_{-\pi/2}^{\pi/2} \frac{\sec^2\theta \, d\theta}{\sec^{2m}\theta \, d\theta} = \int_{-\pi/2}^{\pi/2} \cos^{2m-2}\theta \, d\theta =$$

$$\int_{-\pi/2}^{\pi/2} \frac{(e^{i\theta}+e^{-i\theta})^{2m-2}}{2^{2m-2}} d\theta = \frac{1}{2^{2m-2}} \int_{-\pi/2}^{\pi/2} \sum_{k=0}^{2m-2} (e^{i\theta})^k (e^{-i\theta})^{2m-2-k} \binom{2m-2}{k} d\theta$$

$$= \frac{1}{2^{2m-2}} \sum_{k=0}^{2m-2} \binom{2m-2}{k} \int_{-\pi/2}^{\pi/2} e^{i\theta(2k+2-2m)} d\theta.$$

If $2r$ is even $\neq 0$, $\int_{-\pi/2}^{\pi/2} e^{2ri\theta} d\theta = \frac{e^{2ri\theta}}{2ri}\Big|_{-\pi/2}^{\pi/2} = \frac{1}{2ri}(e^{\pi ri} - e^{-\pi ri}) = 0$.

$$= \frac{1}{2^{2m-2}} \binom{2m-2}{m-1} \int_{-\pi/2}^{\pi/2} e^{i\theta \cdot 0} d\theta = \text{same answer.}$$

2. $\int_{-\infty}^{\infty} \frac{dx}{1+x^{2m}}$ Again, $\deg(1+x^{2m}) = 2m$, $\deg(1)=0$, so if $2m \ge 2$, we can use the residue approach

$$z^{2m}+1=0 \text{ if } z^{2m}=-1=e^{\pi i} \Rightarrow z = e^{\frac{\pi i}{2m}+k\cdot\frac{2\pi i}{2m}} = e^{(2k+1)\cdot\frac{\pi i}{2m}} \quad 0\le k\le 2m-1$$

Let $\zeta = e^{\frac{\pi i}{2m}}$; these zeros are $\zeta, \zeta^3, \dots, \zeta^{2m-1}, \zeta^{2m+1}, \dots, \zeta^{4m-1}$.

Of these, $\zeta, \dots, \zeta^{2m-1}$ are in the upper half plane, because

$\mathrm{Im}(\zeta^r) = \sin r\frac{\pi i}{2m} > 0$ if $0<r<2m$, and <0 if $2m<r<4m$.

If $f(z)=z^{2m}+1$, then $f'(z)=2mz^{2m-1}$, so each pole has order 1 and by our shortcut,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^{2m}} = 2\pi i\left[\frac{1}{2m\zeta^{2m-1}} + \frac{1}{2m(\zeta^3)^{2m-1}}+\dots+\frac{1}{2m(\zeta^{2m-1})^{2m-1}}\right]$$

Technique: If $\alpha^{2m}+1=0$, then $\frac{1}{\alpha^{2m-1}} = \frac{\alpha}{\alpha^{2m}} = -\alpha$.

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^{2m}} = -\frac{2\pi i}{2m}\left[\zeta+\zeta^3+\dots+\zeta^{2m-1}\right]$$

Technique: $\zeta+\zeta^3+\dots+\zeta^{2m-1} = \zeta(1+\zeta^2+\dots+\zeta^{2m-2}) = \zeta\cdot\frac{1-\zeta^{2m}}{1-\zeta^2} = \frac{2\zeta}{1-\zeta^2}$

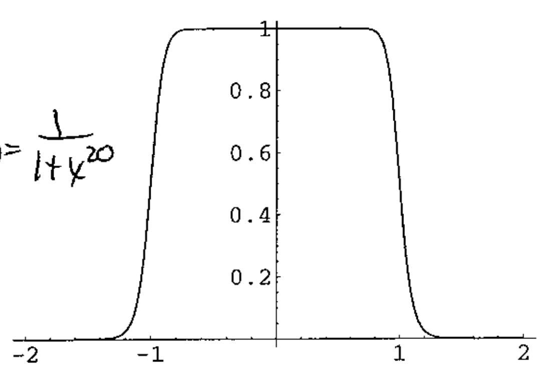
(Remember: technique = trick you've seen)

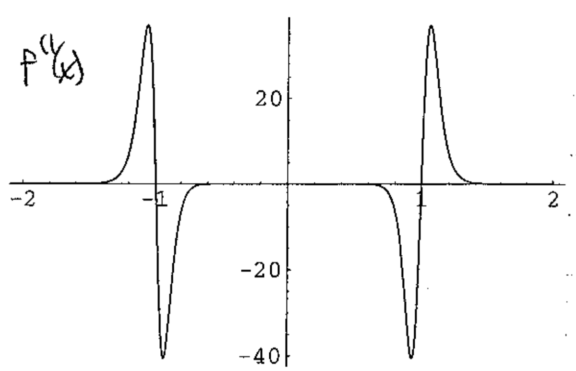
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^{2m}} = -\frac{2\pi i}{2m}\cdot\frac{2\zeta}{1-\zeta^2} = \frac{2\pi i}{2m}\cdot\frac{2\zeta}{\zeta^2-1} = \frac{2\pi}{2m}\,\frac{1}{\frac{\zeta^2-1}{2i\zeta}} = \frac{2\pi}{2m}\cdot\frac{1}{\frac{\zeta-\zeta^{-1}}{2i}}$$

Since $\frac{\zeta-\zeta^{-1}}{2i} = \frac{e^{\frac{\pi i}{2m}}-e^{-\frac{\pi i}{2m}}}{2i} = \sin\left(\frac{\pi}{2m}\right)$, we get the formula

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^{2m}} = \frac{2\pi}{2m}\,\frac{1}{\sin\pi/2m} = 2\cdot\frac{\frac{\pi}{2m}}{\sin\frac{\pi}{2m}}$$ As $m\to\infty$, this $\to 2$

$f(x)=\frac{1}{1+x^{20}}$





Math 448
Extra Notes
11/9/07

1. The binomial theorem.

(i) Fix a (complex) number α, and let $f_\alpha(z) = (1+z)^\alpha$, by which we mean $e^{\alpha \operatorname{Log}(1+z)}$, for $|z|<1$. Then

$$f_\alpha'(z) = e^{\alpha \operatorname{Log}(1+z)} \cdot (\alpha \operatorname{Log}(1+z))' = e^{\alpha \operatorname{Log}(1+z)} \cdot \frac{\alpha}{1+z} = \alpha e^{(\alpha-1)\operatorname{Log}(1+z)}$$

$= \alpha(1+z)^{\alpha-1}$, conveniently enough. An easy (and omitted) induction shows that for $k \geq 1$ $f_\alpha^{(k)}(z) = \alpha(\alpha-1)\cdots(\alpha-(k-1))\cdot(1+z)^{\alpha-k}$

In particular, $f_\alpha^{(k)}(0) = \alpha(\alpha-1)\cdots(\alpha-(k-1))$ and the power series of f_α at $z=0$ yields

$$(1+z)^\alpha = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-(k-1))}{1\cdot 2 \cdots k} z^k \qquad (*)$$

Using the ratio test, the ratio of the k+1st to kth terms above in $(*)$ is $\left|\frac{\alpha-k}{k+1}\right|\cdot|z|$; $\lim_{k\to\infty} \left|\frac{\alpha-k}{k+1}\right|\cdot|z| = |z|$ for any α, hence this series does converge for $|z|<1$, as desired.

(ii) When $\alpha \in \mathbb{N}$, we get the usual polynomial. When $\alpha = \frac{1}{2}$

$$\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\cdots\left(-\frac{2k-3}{2}\right)}{1\cdot 2\cdots k} = \frac{1}{k!}\cdot\frac{1}{2^k}(-1)^{k-1}\cdot 1\cdot 3\cdot 5\cdots(2k-3)$$

by an old idea, $1\cdot 3\cdot 5\cdots(2k-3) = 1\cdot\frac{2}{2}\cdot 3\cdot\frac{4}{4}\cdots 2k-3\cdot\frac{2k-2}{2k-2} = \frac{(2k-2)!}{\underbrace{2\cdot 4\cdots 2k-2}_{2\cdot 1\cdot 2\cdot 2\cdots 2\cdot(k-1)}}$

$$= \frac{(2k-2)!}{2^{k-1}(k-1)!}, \text{ so}$$

$$(1+z)^{1/2} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!}\cdot\frac{(2k-2)!}{2^{2k-1}\cdot(k-1)!}\cdot z^k$$

$$= 1 + \sum_{k=1}^{\infty} -\frac{2}{k}\binom{2k-2}{k-1}\left(\frac{-z}{4}\right)^k$$

(or, replacing z by $-4z$,

$$(1-4z)^{1/2} = 1 \quad \sum_{k=1}^{\infty} \frac{2}{k}\binom{2k-2}{k-1} z^k.$$

2. The homework problem.

I asked you to compute $\int_0^\infty \frac{x^2}{(x^2+9)^m}dx$ for $m=2,3$ by the residue method, which is valid if $2m \geq 2+2$, or $m=2$, and it occurred to me that this could illustrate generating functions.

For $k \geq 0$, let

$$a_k = \int_0^\infty \frac{x^2}{(x^2+9)^{2+k}}dx$$

Since $\frac{x^2}{(x^2+9)^{\cancel{2+k}+1}} = \frac{1}{x^2+9}\cdot\frac{x^2}{(x^2+9)^{2+k}} \leq \frac{1}{9}\cdot\frac{x^2}{(x^2+9)^{2+k}}$,

I knew that $a_{k+1} \leq \frac{1}{9}a_k$, so $F(t) := \sum_{k=0}^\infty a_k t^k$ will converge if $|t|<9$.

We now write:

$$F(t) = \sum_{k=0}^\infty a_k t^k = \sum_{k=0}^\infty \left(\int_0^\infty \frac{x^2}{(x^2+9)^{2+k}}dx\right)t^k$$

$$= \int_0^\infty \left(\sum_{k=0}^\infty \frac{x^2\cdot t^k}{(x^2+9)^{2+k}}\right)dx = \int_0^\infty \frac{x^2}{(x^2+9)^2}\left(\sum_{k=0}^\infty \frac{t^k}{(x^2+9)^k}\right)dx$$

$$= \int_0^\infty \left(\frac{x^2}{(x^2+9)^2}\cdot\frac{1}{1-\frac{t}{x^2+9}}\right)dx = \int_0^\infty \frac{x^2}{(x^2+9)(x^2+9-t)}dx$$

Fine points: Ordinarily, one has to be careful about interchanging summation and integration. However, if the function is always positive, this can be done safely, with the understanding that it's possible you'll get $\infty = \infty$. Consult an advanced course for details!

$F(0) = \int_0^\infty \frac{x^2}{(x^2+9)^2}dx = \frac{\pi}{12}$ (as we've seen)

If $t \neq 0$, and real just to make our computations easier,

$$F(t) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)(x^2+9-t)}\,dx = \frac{1}{2}\cdot 2\pi i \cdot \sum_{\substack{\text{poles in}\\ \text{upper}\\ \text{halfplu.}}} \text{Res}\,(f(z), \text{poles}) \qquad f(z) = \frac{z^2}{(z^2+9)(z^2+9-t)}$$

Now f has (simple) poles at $\pm 3i$ ad $\pm\sqrt{9-t}\,i$ (note $|t|<9$). (Recall that $t \neq 0$, so those are all destinct. Using the usual rules,

$$F(t) = \pi i \cdot \left[\frac{\frac{(3i)^2}{(3i)^2+9-t}}{2\cdot 3i} + \frac{\frac{(\sqrt{9-t}\,i)^2}{(\sqrt{9-t}\,i)^2+9}}{2\cdot\sqrt{9-t}\,i} \right]$$

$$= \pi i \cdot \left[\frac{-9/-t}{6i} + \frac{-\frac{9-t}{t}}{2\sqrt{9-t}\,i} \right] = \frac{\pi}{t}\cdot\left[\frac{3}{2} - \frac{\sqrt{9-t}}{2}\right]$$

$$= \frac{3\pi}{2}\cdot\left[\frac{1-\sqrt{1-\frac{t}{9}}}{t}\right] \quad \text{after some algebra}$$

By p.1, $1-\sqrt{1-\frac{t}{9}} = 1 - \left[1 + \sum_{k=1}^{\infty} \frac{(-2)}{k}\binom{2k-2}{k-1}\left(\frac{t}{36}\right)^k\right]$

$$= \sum_{k=1}^{\infty} \frac{2}{k}\binom{2k-2}{k-1}\left(\frac{t}{36}\right)^k$$

So $F(t) = \frac{3\pi}{2t}\cdot\sum_{k=1}^{\infty} \frac{2}{k}\binom{2k-2}{k-1}\frac{t^k}{36^k} =$

$$= 3\pi \sum_{k=1}^{\infty} \frac{1}{k}\cdot\binom{2k-2}{k-1}\cdot\frac{t^{k-1}}{36^k}$$

(shift indices) $= 3\pi \sum_{k=0}^{\infty} \frac{1}{k+1}\binom{2k}{k}\frac{t^k}{36^{k+1}} = \sum_{k=0}^{\infty} a_k t^k$

Hence $\int_0^{\infty} \frac{x^2}{(x^2+9)^{2+k}}\,dx = a_k = \frac{3\pi}{k+1}\binom{2k}{k}\frac{1}{36^{k+1}}$

Eg. $a_0 = \frac{3\pi}{1}\cdot\binom{0}{0}\cdot\frac{1}{36} = \frac{\pi}{12}$ $\quad a_2 = \frac{\pi}{7776}$, etc…

$a_1 = \frac{3\pi}{2}\binom{2}{1}\cdot\frac{1}{1296} = \frac{\pi}{432}$

as you saw

Mathematica gives the mysterious

$$a_k = \frac{1}{12}\cdot\frac{1}{9^k}\cdot\frac{\Gamma(k+\frac{1}{2})\sqrt{\pi}}{\Gamma(k+2)}$$

which is equivalent, but that's another story