Math 448
HW1
Due 8/31/07

1.1-1a $z=1+2i,\ w=2-i,\ \zeta=4+3i$

$z+3w = 1+2i+6-3i = \underline{7-i}$

1b. $-2w+\bar{\zeta} = -2(2-i)+(4-3i) = -4+2i+4-3i = \underline{-i}$

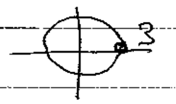
1d. $w^3+w = (2-i)^3+2-i$
$= 8-12i+6i^2-i^3+2-i$
$= 8-12i-6+i+2-i = \underline{4-12i}$

1e. $Re(\zeta^{-1}) = Re\,\frac{1}{4+3i} = Re\,\frac{4-3i}{25} = \frac{4}{25}$

1f. $\frac{w}{z} = \frac{2-i}{1+2i}\cdot\frac{1-2i}{1-2i} = \frac{2-2-i-4i}{5} = \underline{-i}$

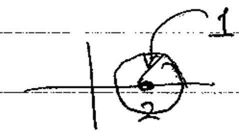
1.1-3a

$|w|=3$

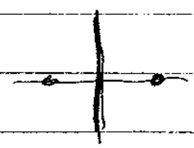


3b.

$|w-2|=1$



3d. $|w+2|=|w-2|$ means the distance from w to 2 and the distance from w to -2 are equal



This gives the perp. bisector

Or. Let $w=u+iv$

$|w+2|^2 = |(u+2)+iv|^2$
$= (u+2)^2+v^2$

$|w-2|^2 = (u-2)^2+v^2$

$(u+2)^2+v^2 = (u-2)^2+v^2$
$\Rightarrow 8u=0 \Rightarrow u=0.$

3f. $Re((1-i)\bar{z}) = 0.$

(i) Let $z = x+iy$, $\bar{z} = x-iy$

$(1-i)\bar{z} = (1-i)(x-iy) = x-y-i(x+y)$

$Re((1-i)\bar{z}) = x-y=0 \iff x=y.$

(ii) Let $z=re^{i\theta}$, $\bar{z}=re^{-i\theta}$

$1-i = \sqrt{2}e^{-i\frac{\pi}{4}}$, so

$(1-i)\bar{z} = \sqrt{2}re^{-i(\theta+\frac{\pi}{4})}$

$Re((1-i)\bar{z}) = \sqrt{2}r\cos(-(\theta+\frac{\pi}{4}))$

so we want $\cos(\theta+\frac{\pi}{4})=0$, i.e., $\theta=\frac{\pi}{4}, \frac{5\pi}{4}$

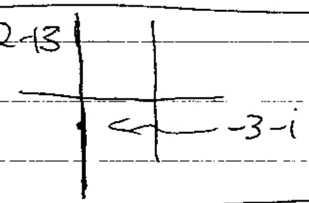
1.2-3

$Re[(4+i)z+6]=0$

$(4+i)(x+iy) = 4x-y+i(x+4y)$

so $4x-y+6=0$ — a line $y=4x+6$

1.2-13



This set is $\{z: Re\,z=-3\}$

1.2-23 $z^5=1$ $z=re^{i\theta}$

$r^5e^{5i\theta}=1 \Rightarrow r^5=1$ $e^{5i\theta}=1$

so $r=1$, $5i\theta=2\pi ki$, $\theta=\frac{2\pi k}{5}$. The distinct solutions are
$1, e^{\frac{2\pi i}{5}}, e^{\frac{4\pi i}{5}}, e^{\frac{6\pi i}{5}}, e^{\frac{8\pi i}{5}}$

~~Wait. I misread #23~~

$z^5=i$ $z=re^{i\theta}$

$r^5e^{5i\theta}=i \Rightarrow r^5=1$ $e^{5i\theta}=e^{\frac{\pi i}{2}}$

so $5\theta=\frac{\pi}{2}+2\pi ki$, $\theta=\frac{\pi}{10}+\frac{2\pi ki}{5}$

multiply the answers above by $e^{\frac{\pi i}{10}}$

1.§1.1-6

(a) $\sqrt{3}e^{\frac{\pi i}{4}} = \sqrt{3}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) = \frac{\sqrt{3}}{\sqrt{2}}+\frac{\sqrt{3}}{\sqrt{2}}i$

(b) $\frac{1}{\sqrt{2}}e^{\pi i} = \frac{1}{\sqrt{2}}(-1+0i) = -\frac{1}{\sqrt{2}}$

(c) $4e^{-\frac{\pi i}{2}} = 4(0-i) = -4i$

(d) $2e^{-\frac{\pi i}{4}} = 2\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right) = \sqrt{2}-\sqrt{2}i$

(e) $1e^{4\pi i} = 1(1+0i) = 1$

(f) $\sqrt{2}e^{\frac{9\pi i}{4}} = \sqrt{2}e^{\frac{\pi i}{4}} = \sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) = 1+i$

2.§1.1-8 Let $z=x+iy$, $|z|^2=x^2+y^2$

A is real, so $z+A = x+A+iy$

and $|z+A|^2 = (x+A)^2+y^2$

$Re(Az) = Re(Ax+iAy) = Ax$

$x^2+y^2+A^2 = (x+A)^2+y^2-2Ax$ ✓

2 (cont'd) $B = u + iv$, $z = x + iy$

$|z|^2 = x^2 + y^2 \quad |B|^2 = u^2 + v^2$

$Bz = ux - vy + i(vx + uy)$

$\mathrm{Re}(Bz) = ux - vy$

$z + \bar{B} = x + u + i(y - v)$

$|z + \bar{B}|^2 = (x+u)^2 + (y-v)^2$

Check the algebra:

$x^2 + y^2 + 2(ux - vy) \stackrel{?}{=}$

$(x+u)^2 + (y-v)^2 - (u^2 + v^2)$

$= x^2 + 2ux + u^2 + y^2 - 2yv + v^2 - u^2 - v^2$ ✓

3. §1.2 6. $|z - i| = \mathrm{Re}\, z$

Let $z = x + iy$, $z - i = x + i(y-1)$

$\sqrt{x^2 + (y-1)^2} = x \quad (\geq 0).$

Square and cancel:

$x^2 + (y-1)^2 = x^2 \Rightarrow (y-1)^2 = 0$

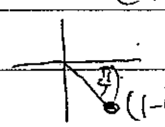
$\Rightarrow y = 1$ so $z = x + 1 \cdot i$

But, $|z - i| \geq 0$, so x must be ≥ 0

I will concede that this is tricky

4. §1.2-29

$(z+1)^4 = 1 - i = \sqrt{2}\, e^{-\frac{\pi i}{4}}$



Let $w = z + 1$

$w^4 = \sqrt{2}\, e^{-\frac{\pi i}{4}}$

$\Rightarrow w = (\sqrt{2})^{\frac{1}{4}} e^{\frac{1}{4}\left(-\frac{\pi}{4}\right)} e^{\frac{2\pi i k}{4}}$ $\quad k = 0, 1, 2, 3$

ie, $w = 2^{\frac{1}{8}} e^{-\frac{\pi i}{16}} * \{1, i, -1, -i\}$

and $z = w - 1$, so

$z = -1 + 2^{\frac{1}{8}} e^{-\frac{\pi i}{16}} i^k \quad k = 0, 1, 2, 3.$

There are many correct ways to write these

5. $z^3 = -8 = 8e^{\pi i}$

$z = re^{i\theta} \Rightarrow r^3 e^{3i\theta} = 8e^{\pi i}$

$r^3 = 8 \Rightarrow r = 2 \quad e^{3i\theta} = e^{\pi i}$

$\Rightarrow 3\theta = \pi + 2\pi k \quad k = 0, 1, 2.$

$\theta = \frac{\pi}{3} + \frac{2\pi k}{3}$

So $z = 2e^{\frac{\pi}{3} i}, 2e^{\pi i}, 2e^{\frac{5\pi i}{3}}$

Using trig: $e^{\frac{\pi}{3} i} = \frac{1}{2} + \frac{\sqrt{3} i}{2}$

$e^{\frac{5\pi i}{3}} = \frac{1}{2} - \frac{\sqrt{3} i}{2}$,

So $z = 1 + \sqrt{3} i, -2, 1 - \sqrt{3} i$

6. $2 + 2i = 2\sqrt{2}\, \frac{1+i}{\sqrt{2}} = 2\sqrt{2}\, e^{\frac{\pi i}{4}}$

$(2+2i)^{2007} = \left(2^{3/2} e^{\frac{\pi i}{4}}\right)^{2007}$

$= \left(2^{3010}\sqrt{2}\right) \cdot e^{\pi i \left(501 + \frac{3}{4}\right)}$ $\quad 501\frac{3}{4} = 502 - \frac{1}{4}$

$= \left(2^{3010}\sqrt{2}\right) \cdot e^{-\frac{\pi i}{4}}$

$= 2^{3010}\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$

$= 2^{3010}(1 - i)$

7. $z^3 = \bar{z}$

If $z = re^{i\theta}$, $z^3 = r^3 e^{3i\theta}$, $\bar{z} = re^{-i\theta}$

$r^3 e^{3i\theta} = re^{-i\theta} \Rightarrow r^3 = r, \; e^{3i\theta} = e^{-i\theta}$

$r^3 = r \Leftrightarrow r = 0, 1$ (since $r \geq 0$)

$e^{3i\theta} = e^{-i\theta} \Rightarrow e^{4i\theta} = 1 \Rightarrow e^{i\theta} \in \{1, -1, i, -i\}$

If $r = 0$, $e^{i\theta}$ doesn't matter, so

$z = 0, 1, i, -1$ or $-i$

8. §1.1-20 Suppose $|\lambda|^2 C - 2\mathrm{Re}(\bar{\lambda} A) + B \geq 0$ for real numbers $B, C \geq 0$ and a fixed complex number A, and all complex λ.

If $C = 0$, then $B - 2\mathrm{Re}(\bar{\lambda} A) \geq 0$

Choose $\lambda = t \cdot A$, where $t > 0$ is real

Then $\bar{\lambda} A = t\bar{A} \cdot A = t|A|^2$, so

$B - t|A|^2 \geq 0$ for all real t.

This implies that $|A|^2 = 0$, ie $A = 0$

8. cont'd.

If $C \neq 0$, let $\lambda = \frac{A}{C}$. Then

$|\lambda|^2 C - 2\,\mathrm{Re}(\bar{\lambda} A) + B$

$= \frac{|A|^2}{C^2} \cdot C - 2\,\mathrm{Re}\left(\frac{\bar{A}A}{C}\right) + B$

$= \frac{|A|^2}{C} - \frac{2|A|^2}{C} + B = \frac{BC - |A|^2}{C}$

≥ 0, so that $BC \geq |A|^2$

21. Cauchy-Schwarz

$0 \leq \sum_{j=1}^{n} |a_j - \lambda b_j|^2$

$= \sum_{j=1}^{n} |a_j|^2 - 2\,\mathrm{Re} \sum a_j \bar{b}_j \lambda$

$+ \lambda^2 \sum_{j=1}^{n} |b_j|^2$

which fits in above

with $C = \sum |b_j|^2$, $B = \sum_{j=1}^{n} |a_j|^2$

and $A = \sum_{j=1}^{n} a_j \bar{b}_j$.

If $C = 0$, then $A = 0$, and $BC \geq |A|^2$ trivially

If $C \neq 0$, then $BC \geq |A|^2$ (all $b_j = 0$)

$\therefore \left(\sum_{j=1}^{n} |a_j|^2\right)\left(\sum_{j=1}^{n} |b_j|^2\right)$

$\geq \left(\sum_{j=1}^{n} a_j \bar{b}_j\right)^2$.

---

9. §1.2 - 36, 38 $L = \{(\cos\theta\, t, \sin\theta\, t) : t \in \mathbb{R}\}$

36. $z = t(\cos\theta + i\sin\theta) = te^{i\theta}$

$\frac{1}{z} = \frac{1}{te^{i\theta}} = \frac{1}{t} e^{-i\theta}$

$(t \neq 0)$ $\quad = \frac{1}{t}(\cos\theta - i\sin\theta)$

The slopes are negatives of each other (or if the line is vertical, the same).

Subtle point: $re^{i\theta}$, $r \geq 0$ gives a ray, not a line

38. $|z - r| = r \Rightarrow z - r = r\cos\theta + r\sin\theta$

$z = (r + r\cos\theta) + r\cdot\sin\theta\, i$

$\frac{1}{z} = \frac{(r + r\cos\theta) - r\sin\theta\, i}{(r + r\cos\theta)^2 + (r\sin\theta)^2}$

$(r + r\cos\theta)^2 + (r\sin\theta)^2$

$= r^2 (1 + 2\cos\theta + \cos^2\theta + \sin^2\theta)$

$= 2r^2(1 + \cos\theta)$

so $\frac{1}{z} = \frac{r(1+\cos\theta) - r\cdot\sin\theta\, i}{2r^2(1+\cos\theta)}$

$= \frac{1}{2r} - \frac{1}{2r} \cdot \frac{\sin\theta}{1+\cos\theta} i$

Trig fans will recognize that if $z = \rho e^{i\alpha}$ then

$\tan\alpha = \frac{\sin\theta}{1+\cos\theta} = \tan\frac{\theta}{2}$ so $\alpha = \frac{\theta}{2}$

As we'll see later

$r + re^{i\theta} = r(1 + e^{i\theta})$

$= re^{i\theta/2}(e^{-i\frac{\theta}{2}} + e^{i\frac{\theta}{2}})$

$= 2r\cos\frac{\theta}{2} \cdot e^{\frac{i\theta}{2}}$.

---

10 a) Let $(3+4i)^n = a_n + ib_n$.

Then $(3+4i)^{n+1} = (3+4i)(3+4i)^n$

$\Rightarrow$

$a_{n+1} + ib_{n+1} = (3+4i)(a_n + ib_n)$

ie $a_{n+1} = 3a_n - 4b_n$

$b_{n+1} = 4a_n + 3b_n$.

b). Clearly, $a_1 = 3$, $b_1 = 4$, so

"$a_n \equiv 3 \bmod 5$, $b_n \equiv 4 \bmod 5$"

is valid for $n = 1$.

If true for n, then

$a_{n+1} = 3a_n - 4b_n \equiv 3\cdot 3 - 4 4$

$\equiv 9 - 16 \equiv -7 \equiv 3 \bmod 5$

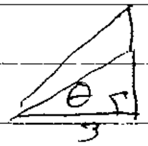
$b_{n+1} = 4a_n + 3b_n \equiv 4\cdot 3 + 3\cdot 4$

$\equiv 12 + 12 \equiv 24 \equiv 4 \bmod 5$

Thus the assertion is proved by induction.

~~c. Your favorite part!~~

Let $\theta = \frac{1}{\pi}\arctan\frac{4}{3}$



Then $3+4i = 5e^{i\theta\pi}$

Suppose $\theta = \frac{k}{\ell}$ is rational.

Then $3+4i = 5e^{i\cdot\frac{k\pi}{\ell}}$

so $(3+4i)^{\ell} = 5^{\ell}e^{ik\pi}$

$= 5^{\ell}$ or -5^{ℓ}

In particular,

$\mathrm{Im}\left((3+4i)^{\ell}\right) = 0$

But

$(3+4i)^{\ell} = a_{\ell} + i\cdot b_{\ell}$

so $b_{\ell} = 0$ and $b_{\ell} \equiv 4 \pmod 5$

a contradiction!

d.

$a_{n+2} = 3a_{n+1} - 4b_{n+1}$

$= 3(3a_n - 4b_n)$

$- 4(4a_n + 3b_n)$

$a_{n+2} = -7a_n - 24b_n$
$a_{n+1} = 3a_n - 4b_n$] (*)

So $a_{n+2} - 6a_{n+1} =$

$-7a_n - 24b_n - 18a_n + 24b_n$

$= -25a_n$

[Here I'm trying to eliminate b_n from (*).]

$a_{n+2} - 6a_{n+1} + 25a_n = 0$

a similar calculation shows that

$b_{n+2} - 6b_{n+1} + 25b_n = 0.$

The polynomial

$x^2 - 6x + 25$ has roots

$x = \frac{6 \pm \sqrt{36-100}}{2} = 3 \pm 4i$

by the binomial theorem and another way to find r, s is this.

$(3+4i)^n = a_n + ib_n$

$(3-4i)^n = a_n - ib_n$

$a_n = \frac{1}{2}\left[(3+4i)^n + (3-4i)^n\right]$

$b_n = \frac{1}{2i}\left[(3+4i)^n - (3-4i)^n\right]$

Consider the equation

(*) $T_{n+2} + rT_{n+1} + sT_n = 0.$

If $T_n = \lambda^n$, then λ^n satisfies (*) for all n

$\iff \lambda^{n+2} + r\cdot\lambda^{n+1} + s\cdot\lambda^n = 0$ for all n

$\iff \lambda^n(\lambda^2 + \lambda\cdot r + s) = 0$ for all n

$\iff \lambda^2 + \lambda r + s = 0$

This is part of the theory of constant coefficient recurrence sequences

A theorem of Niven (using algebraic methods) establishes that $\frac{1}{\pi}\arctan\frac{m}{n}$ is only rational when $\frac{m}{n} \in \{0, \pm 1\}$. . . !