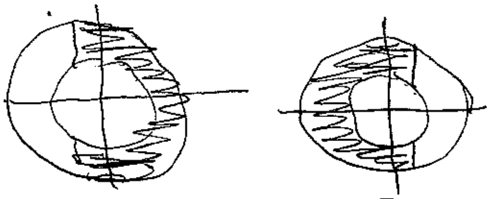
§1.3-15

$\Omega_1 = \{z : 1<|z|<2\} \operatorname{Re} z > -\frac{1}{2}\}$

$\Omega_2 = \{z : 1<|z|<2 \ \& \operatorname{Re} z < \frac{1}{2}\}$

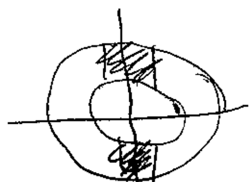
Ω_1 and Ω_2 are each the intersection of two open sets.



Ω_1 Ω_2

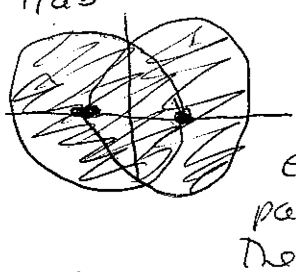
both are clearly connected. $\Omega_1 \cap \Omega_2$ is open, but it's not connected

$\Omega_1 \cap \Omega_2$



1.3-17ab

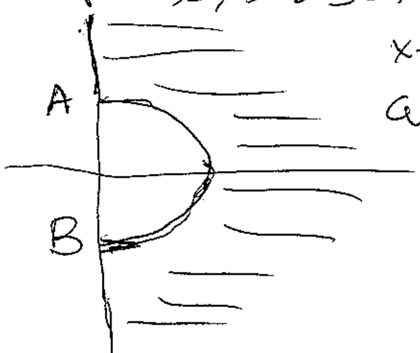
a.



$|z-1|<2$ or $|z-(-1)|<2$

each circle passes through the center of the other, and any point in the intersection, eg $z=0$ "sees" all points, so it's star shaped.

b.



$x+iy : |z|>1$ and $x>0$

Intuitively, not star-shaped but hard to prove. Basically, the only points that see everything near A are $\{x+iy : y \geq 1\}$ and those points that see everything near B are $\{x+iy : y \leq -1\}$

This were topology, I'd expect a better solution.

§1.4-13 $|z|^2 = z\bar{z}$

Math 448
Hw 2
Due 9/7/07

$f(z) = \frac{|z|^2}{z}, z \neq 0.$

$f(z) = \begin{cases} \bar{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

so $\lim_{z\to 0} f(z) = \lim_{z \to 0} \bar{z} = 0.$

§1.4-33 $\sum_{n=1}^{\infty} \left(\frac{2+i}{\sqrt{5}}\right)^n$

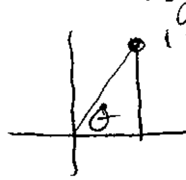
Observe that $\left|\frac{2+i}{\sqrt{5}}\right|^2 = \frac{2^2}{5} + \frac{1^2}{5} = 1,$

so $\left(\frac{2+i}{\sqrt{5}}\right)^n \not\to 0$, so the series diverges

§1.5-1

$e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = \frac{\sqrt{2}}{2} + \frac{i}{2}\sqrt{2}$

§1.5-3 $\log(1+i\sqrt{3})$ $|1+i\sqrt{3}| = \sqrt{1^2+\sqrt{3}^2} = \sqrt{1+3} = \sqrt{4} = 2$



$\theta = \frac{\pi}{3} = 60°.$

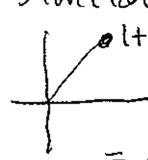
so $\operatorname{Arg}(1+i\sqrt{3}) = \frac{\pi}{3},$

$\arg(1+i\sqrt{3}) = \frac{\pi}{3} + 2\pi k, \ k \in \mathbb{Z}$

$\log(1+i\sqrt{3}) = \ln 2 + i\left(\frac{\pi}{3} + 2\pi k\right), k \in \mathbb{Z}$

§1.5-5 $(1+i)^i = e^{i\log(1+i)}$ $k \in \mathbb{Z}$

Similar to above, $\log(1+i) = \ln\sqrt{2} + i\left(\frac{\pi}{4} + 2\pi k\right)$



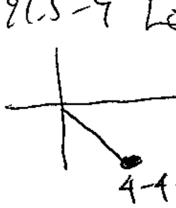
so $(1+i)^i = e^{i(\ln\sqrt{2} + i(\frac{\pi}{4} + 2\pi k))}$ $k \in \mathbb{Z}$

$= e^{i\ln\sqrt{2}} \cdot e^{-\frac{\pi}{4}} e^{-2\pi k}, \ k \in \mathbb{Z}$

$= e^{-\frac{\pi}{4} - 2\pi k}\left(\cos(\ln\sqrt{2}) + i\sin(\ln\sqrt{2})\right)$

§1.5-9 $\operatorname{Log}(4-4i)$ $|4-4i| = \sqrt{16+16} = 4\sqrt{2}$

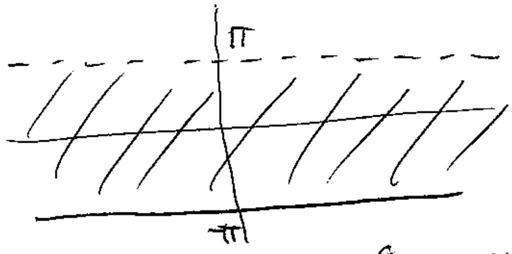
$\operatorname{Arg}(4-4i) = -\frac{\pi}{4}$



$\operatorname{Log}(4-4i) = \ln(4\sqrt{2}) - \frac{\pi}{4}i$

1. §1.3 – 8

$H = \{z = x + iy : -\pi \le y < \pi\}$



(a) The interior is $\{z : -\pi < y < \pi\}$

~~(b)~~ The boundary is $\{z : y = -\pi \text{ or } \pi\}$

(b) This set is neither open nor closed

(c) The interior is connected (actually, it's convex)

---

2. §1.4 – 16

$$f(z) = \begin{cases} \dfrac{z^4-1}{z-i}, & z \neq i \\ 4i, & z = i. \end{cases}$$

$$\frac{z^4-1}{z-i} = \frac{z^4-i^4}{z-i} = z^3 + z^2 i + z i^2 + i^3$$
$$= z^3 + iz^2 - z - i$$

This is a polynomial, so if $z_0 \neq i$, f is continuous at z_0

and $\lim\limits_{z \to i} f(z) = i^3 + i^3 + i^3 + i^3$
$= 4i^3 = -4i$

so f is not continuous at $z = i$.

---

3. §1.4 – 34, 36

34. $\sum\limits_{n=1}^{\infty} \frac{1}{2+i^n}$ $\quad i^n \in \{1, i, -1, -i\}$ so $2 + i^n \in \{3, 2+i, 1, 2-i\}$

we see that $\lim\limits_{n\to\infty} \frac{1}{2+i^n}$ does not exist

and in particular, it is $\neq 0$, so the series diverges

36. $\sum\limits_{n=1}^{\infty} \frac{1}{n^2+i^n}$ $\quad |n^2 + i^n| \ge |n^2| - |i^n|$ by the triangle inequality

so $|n^2 + i^n| \ge n^2 - 1$ and for $n > 1$,

$\left|\frac{1}{n^2+i^n}\right| \le \frac{1}{n^2-1}$.

Since $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges (by your favorite test), the ratio test

and $\lim\limits_{n\to\infty} \dfrac{\frac{1}{n^2-1}}{\frac{1}{n^2}} = \lim\limits_{n\to\infty} \frac{n^2}{n^2-1} = \lim\limits_{n\to\infty} \frac{1}{1-\frac{1}{n^2}} = 1$

imply that $\sum\limits_{n=1}^{\infty} \frac{1}{n^2-1}$ converges. Thus, $\sum\limits_{n=1}^{\infty} \frac{1}{n^2+i^n}$ converges absolutely, so it converges.

---

§1.5 – 6, 10, 12

6. $2^{-1-i} = e^{(-1-i)(\log 2)}$

$|2| = 2$, $\log 2 = \ln 2 + 2\pi k i$ $\quad k \in \mathbb{Z}$

$\arg 2 = 0 + 2\pi k$

$$2^{-1-i} = e^{-(1+i)[\ln 2 + 2\pi k i]}$$
$$= e^{-\ln 2 - \ln 2 i - 2\pi k i + 2\pi k}$$

acceptable answers

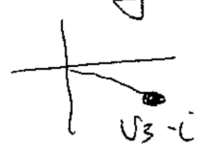
$$= e^{-\ln 2} e^{2\pi k} \cdot e^{-\ln 2 i - 2\pi k i}$$
$$= \frac{1}{2} \cdot e^{2\pi k} \cdot (\cos(\ln 2) - i \sin(\ln 2)) \cdot \underbrace{\left(e^{-2\pi i}\right)^k}_{=1}$$
$$= \frac{1}{2} e^{2\pi k} (\cos(\ln 2) - i \sin(\ln 2)) \quad k \in \mathbb{Z}$$

10. $\mathrm{Log}(-1)$

$|-1| = 1$ $\qquad \mathrm{Arg}(-1) = -\pi$

so $\mathrm{Log}(-1) = \ln 1 - \pi i = -\pi i$

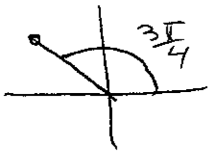
12. $\log(\sqrt{3} - i)$ $\quad |\sqrt{3} - i| = \sqrt{3+1} = 2$



$\mathrm{Arg}(\sqrt{3} - i) = -\frac{\pi}{6}$
$(= -30^\circ)$

so $\log(\sqrt{3} - i) = \ln 2 - \frac{\pi}{6} i + 2\pi k i$
$k \in \mathbb{Z}$

5a. $e^z = -1+i$ $\qquad |z| = \sqrt{2}$



$\operatorname{Arg} z = \frac{3\pi}{4}$

$z = \log(-1+i)$

$= \log\sqrt{2} + i\cdot\left(\frac{3\pi}{4} + 2\pi k\right), \ k\in\mathbb{Z}$

polar. $\quad r = \sqrt{(\log\sqrt{2})^2 + \left(\frac{3\pi}{4}+2\pi k\right)^2}$

$z = re^{i\theta}$ $\quad \theta = \arctan \dfrac{\frac{3\pi}{4}+2\pi k}{\log\sqrt{2}}$

what a silly problem! Whose idea was this?

---

b. $\cos z = -2$

$\dfrac{e^{iz}+e^{-iz}}{2} = -2 \Rightarrow e^{iz}+e^{-iz} = -4$

$(e^{iz})^2 + 4(e^{iz}) + 1 = 0$

$\Rightarrow e^{iz} = -2 \pm \sqrt{3} < 0$

so $iz = \log(-2\pm\sqrt{3})$

$= \ln(2\mp\sqrt{3}) - \pi i + 2\pi k i$

$= \pm\ln(2+\sqrt{3}) + (2k-1)\pi i$

and $z = (2k-1)\pi \pm \ln(2+\sqrt{3})\, i$

$= re^{i\theta}$

$r = \sqrt{((2k-1)\pi)^2 + (\ln(2+\sqrt{3}))^2}$

$\theta = \arctan \dfrac{\pm\ln(2+\sqrt{3})}{(2k-1)\pi} \quad k\in\mathbb{Z}$

ridiculous

---

c. Meant to be a joke.

$\operatorname{Log}(z) = -1 - \pi i \Longrightarrow$

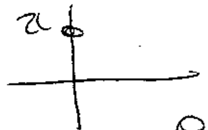
$z = e^{-1-\pi i} = e^{-1}(\cos(-\pi) + i\sin(-\pi))$

$= \underline{-\frac{1}{e}}$

$r = \frac{1}{e} \quad \theta = \pi.$

6. $(2i)^i = e^{i\log(2i)}$

$\log 2i = \ln 2 + \frac{\pi}{2}i + \pi\cdot 2ki \quad k\in\mathbb{Z}$



so

$e^{i\log 2i} = e^{i(\ln 2 + \frac{\pi}{2}i + \pi\cdot 2ki)}$

$= e^{-\frac{\pi}{2}}e^{-2\pi k}(\cos(\ln 2) + i\sin(\ln 2))$

---

7. Suppose $z = e^{i\theta}$ $\quad -\pi \le \theta < \pi$

Then $\operatorname{Arg} z = \theta$.

$z^2 = e^{2i\theta}$ and $-2\pi \le 2\theta < 2\pi$

There are 3 cases

① $-\pi \le 2\theta < \pi$; ie, $-\frac{\pi}{2} \le \theta < \frac{\pi}{2}$.

Then $\operatorname{Arg}(z^2) = 2\theta = 2\operatorname{Arg}(z)$

So $g(z) = 2\theta - 2\theta = 0$.

② $\pi \le 2\theta < 2\pi$; ie $\frac{\pi}{2} \le \theta < \pi$

Then $\operatorname{Arg}(z^2) = 2\theta - 2\pi$ (to get it into the correct interval)

so $\operatorname{Arg}(z^2) - 2\operatorname{Arg} z =$

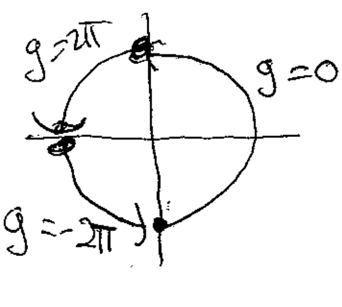
$g(z)$: $2\theta - 2\pi - 2\theta = \underline{-2\pi}$

③ $-2\pi \le 2\theta < -\pi$, ie $-\pi \le \theta < -\frac{\pi}{2}$

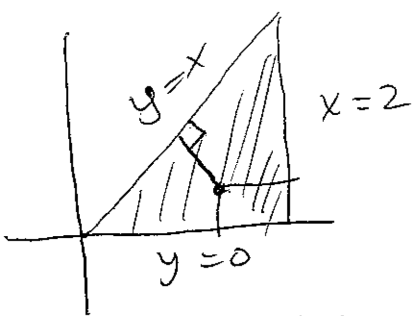
Here, $\operatorname{Arg}(z^2) = 2\theta + 2\pi$, so

$g(z) = 2\theta + 2\pi - 2\theta = \underline{2\pi}$

So the values are $\{0, 2\pi, -2\pi\}$



8. $A = \{ z = x + iy : 0 < y < x < 2$



The distance of (x_0, y_0) to the line $y = 0$ is simply y_0
The distance of (x_0, y_0) to the line $x = 2$ is simply $2 - x_0$
The distance of (x_0, y_0) to the line $y = x$ is the distance to the closest point on the line. Taking the perpendicular, this point is $\left(\frac{x_0+y_0}{2}, \frac{x_0+y_0}{2}\right)$ and the distance is $\sqrt{\left(\frac{x_0-y_0}{2}\right)^2 + \left(\frac{y_0-x_0}{2}\right)^2} = \frac{x_0-y_0}{\sqrt{2}}$

so if we take
$r = \epsilon(x_0, y_0) = \min\left(y_0, 2-x_0, \frac{x_0-y_0}{\sqrt{2}}\right)$
The disk with radius r at (x_0, y_0) will lie in A.

$\epsilon(1.5, .6) = \min\left(.6, .5, \frac{.9}{\sqrt{2}}\right)$
$= .5 \qquad \frac{.9}{\sqrt{2}} \simeq .636$

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9. §1.5-21 $\cosh z = \frac{e^z + e^{-z}}{2} \quad \sinh z = \frac{e^z - e^{-z}}{2}$

Let ~~$A = e^z$, $\frac{1}{A} = e^{-z}$~~

$\cosh^2 z = \frac{e^{2z} + 2 + e^{-2z}}{2^2} \quad \sinh^2 z = \frac{e^{2z} - 2 + e^{-2z}}{2^2}$

a) so $\cosh^2 z - \sinh^2 = \frac{2-(-2)}{4} = 1$

b) $\cos iz = \frac{e^{i(iz)} + e^{-i(iz)}}{2} = \frac{e^{-z} + e^{z}}{2} = \cosh z$

c) $i \sin iz = i \cdot \frac{e^{i(iz)} - e^{-i(iz)}}{2i} = \frac{1}{2}(e^{-z} - e^{z})$
$= -\sinh z \Rightarrow \sinh z = -i \sin iz$.

(iv) $e^z = e^x \cos y + i e^x \sin y$
$e^{-z} = e^{-x} \cos y - i e^{-x} \sin y$

$\cosh z = \frac{e^x + e^{-x}}{2} \cdot \cos y + i \cdot \frac{e^x - e^{-x}}{2} \cdot \sin y$
$\Rightarrow |\cosh^2 z| = \cosh^2 x \cos^2 y + \sinh^2 x \sin^2 y$
$\cosh^2 x = \sinh^2 x + 1$ (real formula)
so $|\cosh^2 z| = (\sinh^2 x + 1)\cos^2 y + \sinh^2 x \sin^2 y$
$= \sinh^2 x (\cos^2 y + \sin^2 y) + \cos^2 y$
$= \sinh^2 x + \cos^2 y$ (many ways to do this)

(v) $\sinh z = \frac{e^x - e^{-x}}{2} \cos y + i \cdot \frac{e^x + e^{-x}}{2} \sin y$
$\Rightarrow |\sinh^2 z| = \sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y$
$= \sinh^2 x \cos^2 y + (1 + \sinh^2 x)\sin^2 y$
$= \sinh^2 x + \sin^2 y$.

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10. First look at $\frac{1}{x + ri}$ for fixed r.

$\frac{1}{x+ri} = \frac{x - ri}{x^2 + r^2} = u + iv$

$u = \frac{x}{x^2+r^2} \quad v = \frac{-r}{x^2+r^2}$

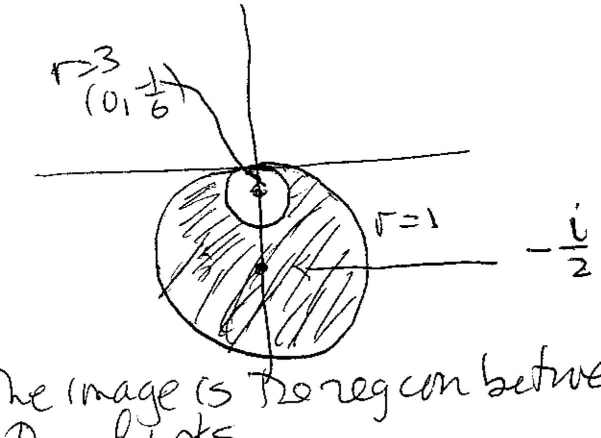
I've already told you that $\frac{1}{z}$ maps lines to circles (and we saw the last homework).

Observe that $u^2 + v^2 = \frac{x^2 + r^2}{(x^2+r^2)^2} = \frac{1}{x^2+y^2}$

so $r(u^2 + v^2) + v = \frac{r - r}{x^2+y^2} = 0$.

$u^2 + v^2 + \frac{1}{r} v = 0$
$u^2 + \left(v + \frac{1}{2r}\right)^2 = \left(\frac{1}{2r}\right)^2$

Circle center $\left(0, -\frac{1}{2r}\right)$ radius $\frac{1}{2r}$



The image is the region between the disks.