The secret lives of polynomial identities

Bruce Reznick
University of Illinois at Urbana-Champaign

~~AMS-MAA Invited Address~~
~~AMS 2010 Spring Southeastern Sectional Meeting~~
~~Lexington, Kentucky, March 27, 2010~~
~~UIUC Number Theory Seminar, April 13, 2010~~
~~UCD Algebra and Discrete Mathematics Seminar, May 21, 2010~~
Math 499, March 2, 2011

“An idea which can be used only once is a trick. If you can use it more than once it becomes a method.” – George Pólya and Gábor Szegö

“An idea which can be used only once is a trick. If you can use it more than once it becomes a method.” – George Pólya and Gábor Szegö

Mathematics is the art of logic and formulas are its poetry.

"An idea which can be used only once is a trick. If you can use it more than once it becomes a method." – George Pólya and Gábor Szegö

Mathematics is the art of logic and formulas are its poetry.

I have always been fascinated by exact formulas, especially finite ones such as polynomial identities.

“An idea which can be used only once is a trick. If you can use it more than once it becomes a method.” – George Pólya and Gábor Szegö

Mathematics is the art of logic and formulas are its poetry.

I have always been fascinated by exact formulas, especially finite ones such as polynomial identities.

They can seem easy and superficial, but the fact that they are true without hypotheses can make mathematicians uncomfortable.

"An idea which can be used only once is a trick. If you can use it more than once it becomes a method." – George Pólya and Gábor Szegö

Mathematics is the art of logic and formulas are its poetry.

I have always been fascinated by exact formulas, especially finite ones such as polynomial identities.

They can seem easy and superficial, but the fact that they are true without hypotheses can make mathematicians uncomfortable.

They don't **need** us.

“An idea which can be used only once is a trick. If you can use it more than once it becomes a method.” – George Pólya and Gábor Szegö

Mathematics is the art of logic and formulas are its poetry.

I have always been fascinated by exact formulas, especially finite ones such as polynomial identities.

They can seem easy and superficial, but the fact that they are true without hypotheses can make mathematicians uncomfortable.

They don’t **need** us.

They come into view unexpectedly, like meteorites on a vast Arctic plain. Once we see them, we must understand that the best ones can signify deep and distant phenomena.

Not all of them are interesting, of course. Sometimes they're just a consequence of linear dependence. For example, who cares that

$$(x+2y)^2 + (2x+3y)^2 + (3x+4y)^2 = 14x^2 + 40xy + 29y^2 \ ?$$

The left hand side has to equal ... *some* binary quadratic form. Identities based on dependence can become interesting if their coefficients have additional properties.

Not all of them are interesting, of course. Sometimes they're just a consequence of linear dependence. For example, who cares that

$$(x+2y)^2+(2x+3y)^2+(3x+4y)^2=14x^2+40xy+29y^2\ ?$$

The left hand side has to equal ... *some* binary quadratic form. Identities based on dependence can become interesting if their coefficients have additional properties.

For example, the binomial theorem and the formula for the n-th difference:

$$\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k=(x+y)^n$$

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(x+ky)^n=n!\ y^n.$$

Not all of them are interesting, of course. Sometimes they're just a consequence of linear dependence. For example, who cares that

$$(x+2y)^2+(2x+3y)^2+(3x+4y)^2 = 14x^2+40xy+29y^2 \ ?$$

The left hand side has to equal ... *some* binary quadratic form. Identities based on dependence can become interesting if their coefficients have additional properties.

For example, the binomial theorem and the formula for the n-th difference:

$$\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k = (x+y)^n$$

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(x+ky)^n = n!\ y^n.$$

Identities are also interesting if there are fewer summands than you'd expect.

What do we want from a “good” polynomial identity?

What do we want from a "good" polynomial identity?

"Astonish me!" – Sergei Diaghilev to Jean Cocteau, on what he wanted in the libretto to the ballet "Parade".

What do we want from a "good" polynomial identity?

"Astonish me!" – Sergei Diaghilev to Jean Cocteau, on what he wanted in the libretto to the ballet "Parade".

This talk consists of stories about some semi-astonishing identities: their deeper meanings and how they can be derived.

The first identity must have its roots in 19th century mathematics, although in this explicit form, I've only been able to trace it back to the mid 1950s. It's one of a family, and it's not accidental that $(1^2 + (\sqrt{3})^2)^5 = 2^{10} = 1024$:

$$\begin{gathered} 1024x^{10} + 1024y^{10} + (x + \sqrt{3}\ y)^{10} + (x - \sqrt{3}\ y)^{10} \\ +(\sqrt{3}\ x + y)^{10} + (\sqrt{3}\ x - y)^{10} = 1512(x^2 + y^2)^5 \end{gathered} \tag{1}$$

The first identity must have its roots in 19th century mathematics, although in this explicit form, I've only been able to trace it back to the mid 1950s. It's one of a family, and it's not accidental that $(1^2 + (\sqrt{3})^2)^5 = 2^{10} = 1024$:

$$\begin{gathered} 1024x^{10} + 1024y^{10} + (x + \sqrt{3}\ y)^{10} + (x - \sqrt{3}\ y)^{10} \\ +(\sqrt{3}\ x + y)^{10} + (\sqrt{3}\ x - y)^{10} = 1512(x^2 + y^2)^5 \end{gathered} \tag{1}$$

The story of (1) and (4) and their generalizations runs through at least number theory, numerical analysis, ~~functional analysis~~ and combinatorics.

The second identity is very old; it goes back to Viète:

$$x^3 + y^3 = \left(\frac{x^4 + 2xy^3}{x^3 - y^3}\right)^3 + \left(\frac{y^4 + 2x^3y}{y^3 - x^3}\right)^3, \qquad (2)$$

The second identity is very old; it goes back to Viète:

$$x^3 + y^3 = \left(\frac{x^4 + 2xy^3}{x^3 - y^3}\right)^3 + \left(\frac{y^4 + 2x^3y}{y^3 - x^3}\right)^3, \tag{2}$$

This is used to show that a sum of two cubes of rational numbers can usually be so expressed in infinitely many ways. For example:

$$2^3 + 1^3 = \left(\frac{20}{7}\right)^3 + \left(-\frac{17}{7}\right)^3 = \left(-\frac{36520}{90391}\right)^3 + \left(\frac{188479}{90391}\right)^3 = \dots$$

The second identity is very old; it goes back to Viète:

$$x^3 + y^3 = \left(\frac{x^4 + 2xy^3}{x^3 - y^3}\right)^3 + \left(\frac{y^4 + 2x^3y}{y^3 - x^3}\right)^3, \qquad (2)$$

This is used to show that a sum of two cubes of rational numbers can usually be so expressed in infinitely many ways. For example:

$$2^3 + 1^3 = \left(\frac{20}{7}\right)^3 + \left(-\frac{17}{7}\right)^3 = \left(-\frac{36520}{90391}\right)^3 + \left(\frac{188479}{90391}\right)^3 = \dots$$

The story here is a description of all homogeneous solutions to

$$x^3 + y^3 = p^3(x,y) + q^3(x,y), \qquad p, q \in \mathbb{C}(x,y).$$

Viète's derivation of his identity, by the way, is formally identical to a common technique in the modern study of elliptic curves.

The third identity was independently found by Desboves (1880) and Elkies (1995):

$$\begin{gathered}(x^2+\sqrt{2}\ x\ y-y^2)^5+(i\ x^2-\sqrt{2}\ x\ y+i\ y^2)^5+\\(-x^2+\sqrt{2}\ x\ y+y^2)^5+(-i\ x^2-\sqrt{2}\ x\ y-i\ y^2)^5=0.\end{gathered}\tag{3}$$

The third identity was independently found by Desboves (1880) and Elkies (1995):

$$(x^2 + \sqrt{2}\, x\, y - y^2)^5 + (i\, x^2 - \sqrt{2}\, x\, y + i\, y^2)^5 + \\ (-x^2 + \sqrt{2}\, x\, y + y^2)^5 + (-i\, x^2 - \sqrt{2}\, x\, y - i\, y^2)^5 = 0. \tag{3}$$

It was first discovered by observing that

$$\sum_{k=0}^{3} (i^k x^2 + i^{2k} a\, x\, y + i^{3k} y^2)^5 = 40a(a^2+2)(x^7y^3 + x^3y^7),$$

and then setting $a = \sqrt{-2}$ and $y = i\ y$. But why $\sqrt{-2}$? The full story ultimately depends on Newton's Theorem on symmetric polynomials. Commutative algebra and algebraic geometry also play a role.

The fourth identity was used by Liouville to show that every positive integer is a sum of at most 53 4th powers of integers:

$$\sum_{1\le i<j\le 4}\left((x_i+x_j)^4+(x_i-x_j)^4\right)=6(x_1^2+x_2^2+x_3^2+x_4^2)^2. \tag{4}$$

The fourth identity was used by Liouville to show that every positive integer is a sum of at most 53 4th powers of integers:

$$\sum_{1\le i<j\le 4} \left((x_i + x_j)^4 + (x_i - x_j)^4\right) = 6(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2. \qquad (4)$$

Many similar and more complicated formulas were found in the late 19th century, until Hilbert showed that they must exist in all degrees.

As one indication of the geometric and combinatorial significance of such identities, if you take the the coordinates of the coefficients of the $2\binom{4}{2}$ linear forms in this identity, together with their antipodes, you get the 24 points $(\pm 1, \pm 1, 0, 0)$ and their permutations.

The fourth identity was used by Liouville to show that every positive integer is a sum of at most 53 4th powers of integers:

$$\sum_{1\le i<j\le 4}\left((x_i+x_j)^4+(x_i-x_j)^4\right)=6(x_1^2+x_2^2+x_3^2+x_4^2)^2. \qquad (4)$$

Many similar and more complicated formulas were found in the late 19th century, until Hilbert showed that they must exist in all degrees.
As one indication of the geometric and combinatorial significance of such identities, if you take the the coordinates of the coefficients of the $2\binom{4}{2}$ linear forms in this identity, together with their antipodes, you get the 24 points $(\pm1,\pm1,0,0)$ and their permutations.

These are the vertices of a regular polytope in $\mathbb{R}^4$ called the *24-cell*.

Psst. He's too busy to notice I'm here

One idea we'll use a lot is, I hope, fairly familiar. Suppose $2 \le d \in \mathbb{N}$. Let

$$\zeta_d = e^{\frac{2\pi i}{d}} = \cos\left(\frac{2\pi}{d}\right) + i \sin\left(\frac{2\pi}{d}\right)$$

denote a primitive d-th root of unity: the solutions to the equation $z^d = 1$ are given by $\{\ \zeta_d^k\ :\ 0 \le k \le d-1\}$.

Psst. He's too busy to notice I'm here

One idea we'll use a lot is, I hope, fairly familiar. Suppose $2 \le d \in \mathbb{N}$. Let

$$\zeta_d = e^{\frac{2\pi i}{d}} = \cos\left(\frac{2\pi}{d}\right) + i\sin\left(\frac{2\pi}{d}\right)$$

denote a primitive *d*-th root of unity: the solutions to the equation $z^d = 1$ are given by $\{\, \zeta_d^k \;:\; 0 \le k \le d-1\}$. Since the sum below is a finite geometric progression, it is easy to see that

Lemma

$$\sum_{r=0}^{d-1} \left(\zeta_d^k\right)^r = \begin{cases} d, & \textit{if } d \mid k; \\ 0, & \textit{otherwise.} \end{cases}$$

We'll use this lemma in sums of polynomials "synched" with powers of ζ_d, so that "most" monomials vanish automatically.

Randy's hoping for some topology today. Fat chance.

Let's look at the first identity again and pull out a factor of 2^{10}:

$$1024x^{10} + 1024y^{10} + (x + \sqrt{3}\ y)^{10} + (x - \sqrt{3}\ y)^{10}$$
$$+(\sqrt{3}\ x + y)^{10} + (\sqrt{3}\ x - y)^{10} = 1512(x^2 + y^2)^5.$$

Randy's hoping for some topology today. Fat chance.

Let's look at the first identity again and pull out a factor of 2^{10}:

$$1024x^{10} + 1024y^{10} + (x + \sqrt{3}\ y)^{10} + (x - \sqrt{3}\ y)^{10}$$
$$+(\sqrt{3}\ x + y)^{10} + (\sqrt{3}\ x - y)^{10} = 1512(x^2 + y^2)^5.$$

becomes

$$x^{10} + y^{10} + \left(\frac{1}{2}\ x + \frac{\sqrt{3}}{2}\ y\right)^{10} + \left(\frac{1}{2}\ x - \frac{\sqrt{3}}{2}\ y\right)^{10}$$
$$+ \left(\frac{\sqrt{3}}{2}\ x + \frac{1}{2}\ y\right)^{10} + \left(\frac{\sqrt{3}}{2}\ x - \frac{1}{2}\ y\right)^{10} = \frac{189}{128}\left(x^2 + y^2\right)^5.$$

Randy's hoping for some topology today. Fat chance.

Let's look at the first identity again and pull out a factor of 2^{10}:

$$1024x^{10} + 1024y^{10} + (x + \sqrt{3}\, y)^{10} + (x - \sqrt{3}\, y)^{10}$$
$$+(\sqrt{3}\, x + y)^{10} + (\sqrt{3}\, x - y)^{10} = 1512(x^2 + y^2)^5.$$

becomes

$$x^{10} + y^{10} + \left(\frac{1}{2}\, x + \frac{\sqrt{3}}{2}\, y\right)^{10} + \left(\frac{1}{2}\, x - \frac{\sqrt{3}}{2}\, y\right)^{10}$$
$$+\left(\frac{\sqrt{3}}{2}\, x + \frac{1}{2}\, y\right)^{10} + \left(\frac{\sqrt{3}}{2}\, x - \frac{1}{2}\, y\right)^{10} = \frac{189}{128}\left(x^2 + y^2\right)^5.$$

It's looking better already. You may recognize this as

Fasten your seat belts

$$\sum_{j=0}^{5} \left(\cos\left(\frac{j\pi}{6}\right) x + \sin\left(\frac{j\pi}{6}\right) y \right)^{10} = \frac{189}{128}(x^2 + y^2)^5.$$

Fasten your seat belts

$$\sum_{j=0}^{5}\left(\cos\left(\frac{j\pi}{6}\right)x+\sin\left(\frac{j\pi}{6}\right)y\right)^{10}=\frac{189}{128}(x^2+y^2)^5.$$

(The first explicit appearance I've found of the underlying general theorem below is in a paper of Avner Friedman (1957).)

Theorem

If $d > r$, then for all θ,

$$\sum_{j=0}^{d-1}\left(\cos\left(\frac{2j\pi}{2d}+\theta\right)x+\sin\left(\frac{2j\pi}{2d}+\theta\right)y\right)^{2r} = \frac{d}{2^{2r}}\binom{2r}{r}(x^2+y^2)^r \tag{5}$$

Dude, you're not supposed to **prove** stuff in a talk!

Taking $d = 6, r = 5$ and $\theta = 0$ in (5) and noting $\frac{6}{2^{10}}\binom{10}{5} = \frac{6\cdot 252}{1024}$ $= \frac{1512}{1024} = \frac{189}{128}$, we get (1).

Dude, you're not supposed to **prove** stuff in a talk!

Taking $d = 6, r = 5$ and $\theta = 0$ in (5) and noting $\frac{6}{2^{10}}\binom{10}{5} = \frac{6 \cdot 252}{1024}$ $= \frac{1512}{1024} = \frac{189}{128}$, we get (1). The fastest proof of the Theorem is to derive it from another formula. Expand the left-hand side below, switch the order of summation and recall that $\zeta_{2d}^{2m} = \zeta_d^m$.

$$\sum_{j=0}^{d-1} (\zeta_{2d}^{j} u + \zeta_{2d}^{-j} v)^{2r}$$

Dude, you're not supposed to **prove** stuff in a talk!

Taking $d = 6, r = 5$ and $\theta = 0$ in (5) and noting $\frac{6}{2^{10}}\binom{10}{5} = \frac{6\cdot 252}{1024}$ $= \frac{1512}{1024} = \frac{189}{128}$, we get (1). The fastest proof of the Theorem is to derive it from another formula. Expand the left-hand side below, switch the order of summation and recall that $\zeta_{2d}^{2m} = \zeta_d^m$.

$$\sum_{j=0}^{d-1}(\zeta_{2d}^{j}u + \zeta_{2d}^{-j}v)^{2r} = \sum_{k=0}^{2r}\binom{2r}{k}\left(\sum_{j=0}^{d-1}\zeta_{2d}^{j(2r-k)+(-j)k}\right)u^{2r-k}v^{k}$$

Dude, you're not supposed to **prove** stuff in a talk!

Taking $d = 6, r = 5$ and $\theta = 0$ in (5) and noting $\frac{6}{2^{10}}\binom{10}{5} = \frac{6 \cdot 252}{1024}$ $= \frac{1512}{1024} = \frac{189}{128}$, we get (1). The fastest proof of the Theorem is to derive it from another formula. Expand the left-hand side below, switch the order of summation and recall that $\zeta_{2d}^{2m} = \zeta_d^m$.

$$\sum_{j=0}^{d-1} (\zeta_{2d}^{j} u + \zeta_{2d}^{-j} v)^{2r} = \sum_{k=0}^{2r} \binom{2r}{k} \left(\sum_{j=0}^{d-1} \zeta_{2d}^{j(2r-k)+(-j)k} \right) u^{2r-k} v^k$$

$$= \sum_{k=0}^{2r} \binom{2r}{k} \left(\sum_{j=0}^{d-1} \left(\zeta_{d}^{r-k} \right)^{j} \right) u^{2r-k} v^k$$

Dude, you're not supposed to **prove** stuff in a talk!

Taking $d = 6, r = 5$ and $\theta = 0$ in (5) and noting $\frac{6}{2^{10}}\binom{10}{5} = \frac{6\cdot 252}{1024}$ $= \frac{1512}{1024} = \frac{189}{128}$, we get (1). The fastest proof of the Theorem is to derive it from another formula. Expand the left-hand side below, switch the order of summation and recall that $\zeta_{2d}^{2m} = \zeta_d^m$.

$$\sum_{j=0}^{d-1}(\zeta_{2d}^{j}u + \zeta_{2d}^{-j}v)^{2r} = \sum_{k=0}^{2r}\binom{2r}{k}\left(\sum_{j=0}^{d-1}\zeta_{2d}^{j(2r-k)+(-j)k}\right)u^{2r-k}v^{k}$$

$$= \sum_{k=0}^{2r}\binom{2r}{k}\left(\sum_{j=0}^{d-1}\left(\zeta_{d}^{r-k}\right)^{j}\right)u^{2r-k}v^{k}$$

As we've seen, the inner sum is zero unless $d \mid r-k$. Since $d > r$, the only multiple of d in $\{-r, -(r-1), \dots, 0, \dots, r-1, r\}$ is 0, corresponding to $k = r$.

He has this thing about 14-th powers. Fair warning.

To recap, we have just seen that

$$\sum_{j=0}^{d-1} (\zeta_{2d}^{j} u + \zeta_{2d}^{-j} v)^{2r} = d \binom{2r}{r} u^r v^r.$$

He has this thing about 14-th powers. Fair warning.

To recap, we have just seen that

$$\sum_{j=0}^{d-1} (\zeta_{2d}^{j} u + \zeta_{2d}^{-j} v)^{2r} = d \binom{2r}{r} u^r v^r.$$

Now let $u = \frac{1}{2} e^{i\theta}(x + \frac{y}{i})$ and $v = \frac{1}{2} e^{-i\theta}(x - \frac{y}{i})$. The substitution $e^{i\theta} = \cos\theta + i \sin\theta$ (θ doesn't have to be real!) gives

He has this thing about 14-th powers. Fair warning.

To recap, we have just seen that

$$\sum_{j=0}^{d-1} (\zeta_{2d}^{j} u + \zeta_{2d}^{-j} v)^{2r} = d \binom{2r}{r} u^r v^r.$$

Now let $u = \frac{1}{2} e^{i\theta}(x + \frac{y}{i})$ and $v = \frac{1}{2} e^{-i\theta}(x - \frac{y}{i})$. The substitution $e^{i\theta} = \cos\theta + i \sin\theta$ (θ doesn't have to be real!) gives

$$\zeta_{2d}^{j} u + \zeta_{2d}^{-j} v = \cos\left(\frac{2j\pi}{2d} + \theta\right) x + \sin\left(\frac{2j\pi}{2d} + \theta\right) y,$$

$$u\ v = \frac{x^2 + y^2}{4}.$$

He has this thing about 14-th powers. Fair warning.

To recap, we have just seen that

$$\sum_{j=0}^{d-1} (\zeta_{2d}^{j} u + \zeta_{2d}^{-j} v)^{2r} = d \binom{2r}{r} u^r v^r.$$

Now let $u = \frac{1}{2} e^{i\theta}(x + \frac{y}{i})$ and $v = \frac{1}{2} e^{-i\theta}(x - \frac{y}{i})$. The substitution $e^{i\theta} = \cos\theta + i\sin\theta$ (θ doesn't have to be real!) gives

$$\zeta_{2d}^{j} u + \zeta_{2d}^{-j} v = \cos\left(\frac{2j\pi}{2d} + \theta\right) x + \sin\left(\frac{2j\pi}{2d} + \theta\right) y,$$

$$u\ v = \frac{x^2 + y^2}{4}.$$

This proves the Theorem.

Told ya

Every 19th century math major knew that $\tan(\frac{\pi}{8}) = \sqrt{2} - 1$, so if we take $r = 7$ and $d = 8$ in (5) and let $\lambda = 338 + 239\sqrt{2}$ and $\alpha = \sqrt{2} - 1$, and do some minor bookkeeping, we get

Told ya

Every 19th century math major knew that $\tan(\frac{\pi}{8}) = \sqrt{2} - 1$, so if we take $r = 7$ and $d = 8$ in (5) and let $\lambda = 338 + 239\sqrt{2}$ and $\alpha = \sqrt{2} - 1$, and do some minor bookkeeping, we get

$$\begin{gathered} 2048x^{14} + 2048y^{14} + 16(x+y)^{14} + 16(x-y)^{14} + \\ \lambda\left((x+\alpha y)^{14} + (x-\alpha y)^{14} + (\alpha x + y)^{14} + (\alpha x - y)^{14}\right) \\ = 3432(x^2+y^2)^7. \end{gathered}$$

Told ya

Every 19th century math major knew that $\tan(\frac{\pi}{8}) = \sqrt{2} - 1$, so if we take $r = 7$ and $d = 8$ in (5) and let $\lambda = 338 + 239\sqrt{2}$ and $\alpha = \sqrt{2} - 1$, and do some minor bookkeeping, we get

$$2048x^{14} + 2048y^{14} + 16(x+y)^{14} + 16(x-y)^{14} +$$
$$\lambda\left((x+\alpha y)^{14} + (x-\alpha y)^{14} + (\alpha x + y)^{14} + (\alpha x - y)^{14}\right)$$
$$= 3432(x^2+y^2)^7.$$

To be sure, if you replace $\{\alpha, \lambda, 2048, 16\}$ with unknown constants and ask Mathematica to solve for them, it will do so, almost instantaneously.

Told ya

Every 19th century math major knew that $\tan(\frac{\pi}{8}) = \sqrt{2} - 1$, so if we take $r = 7$ and $d = 8$ in (5) and let $\lambda = 338 + 239\sqrt{2}$ and $\alpha = \sqrt{2} - 1$, and do some minor bookkeeping, we get

$$\begin{gathered} 2048x^{14} + 2048y^{14} + 16(x+y)^{14} + 16(x-y)^{14} + \\ \lambda\left((x+\alpha y)^{14} + (x-\alpha y)^{14} + (\alpha x + y)^{14} + (\alpha x - y)^{14}\right) \\ = 3432(x^2+y^2)^7. \end{gathered}$$

To be sure, if you replace $\{\alpha, \lambda, 2048, 16\}$ with unknown constants and ask Mathematica to solve for them, it will do so, almost instantaneously.

But it won't know why!

Told ya

Every 19th century math major knew that $\tan(\frac{\pi}{8}) = \sqrt{2} - 1$, so if we take $r = 7$ and $d = 8$ in (5) and let $\lambda = 338 + 239\sqrt{2}$ and $\alpha = \sqrt{2} - 1$, and do some minor bookkeeping, we get

$$2048x^{14} + 2048y^{14} + 16(x+y)^{14} + 16(x-y)^{14} + \\ \lambda\left((x+\alpha y)^{14} + (x-\alpha y)^{14} + (\alpha x + y)^{14} + (\alpha x - y)^{14}\right) \\ = 3432(x^2+y^2)^7.$$

To be sure, if you replace $\{\alpha, \lambda, 2048, 16\}$ with unknown constants and ask Mathematica to solve for them, it will do so, almost instantaneously.

But it won't know why!

Or appreciate just how astonishingly groovy this identity is.

I'm not impressed

There are similar formulas in more variables, and these will be discussed when we get to (4). The main obstacle is this. If you want to put N points evenly on a circle, you want to take the vertices of a regular N-gon. But it is far from obvious how to place N points "evenly" on the surface of S^{n-1}.

I'm not impressed

There are similar formulas in more variables, and these will be discussed when we get to (4). The main obstacle is this. If you want to put N points evenly on a circle, you want to take the vertices of a regular N-gon. But it is far from obvious how to place N points "evenly" on the surface of S^{n-1}. As a taste of things to come, we have the following corollary.

Corollary

If $d > r$, $\theta \in \mathbb{R}$ is arbitrary and $p(x,y)$ is a polynomial with degree $\leq 2r+1$, then

$$\frac{1}{2\pi}\int_0^{2\pi} p(\cos t, \sin t)\ dt$$

$$= \frac{1}{2d}\sum_{j=0}^{2d-1} p\left(\cos\left(\frac{2j\pi}{2d}+\theta\right), \sin\left(\frac{2j\pi}{2d}+\theta\right)\right).$$

Wonder what I'll drink with dinner

In 1591 (or 1593), François Viète published a revolutionary work on algebra which has been translated into English as *The Analytic Art* by T. R. Witmer. Viète's "Zetetic XVIII" is

> *Given two cubes, to find numerically two other cubes the sum of which is equal to the difference between those that are given.*

Wonder what I'll drink with dinner

In 1591 (or 1593), François Viète published a revolutionary work on algebra which has been translated into English as *The Analytic Art* by T. R. Witmer. Viète's "Zetetic XVIII" is

> *Given two cubes, to find numerically two other cubes the sum of which is equal to the difference between those that are given.*

Wonder what I'll drink with dinner

In 1591 (or 1593), François Viète published a revolutionary work on algebra which has been translated into English as *The Analytic Art* by T. R. Witmer. Viète's "Zetetic XVIII" is

> *Given two cubes, to find numerically two other cubes the sum of which is equal to the difference between those that are given.*

I'll quote Viète's proof on the next page. Keep in mind that he was working at the dawn of algebra, when mathematicians were not yet comfortable with negative numbers and the algebraic conventions were very fluid. Viète used vowels as variables and consonants as constants.

Wow, I coulda hadda Viète

"Let the two given cubes be B^3 and D^3, the first to be greater and the second to be smaller. Two other cubes are to be found, the sum of which is equal to $B^3 - D^3$. Let $B - A$ be the root of the first one that is to be found, and let $B^2A/D^2 - D$ be the root of the second. Forming the cubes and comparing them with $B^3 - D^3$, it will be found that $3D^3B/(B^3 + D^3)$ equals A. The root of the first cube to be found, therefore, is $[B(B^3 - 2D^3)]/(B^3 + D^3)$ and of the second is $[D(2B^3 - D^3)]/(B^3 + D^3)$. And the sum of the two cubes of these is equal to $B^3 - D^3$."

Wow, I coulda hadda Viète

"Let the two given cubes be B^3 and D^3, the first to be greater and the second to be smaller. Two other cubes are to be found, the sum of which is equal to $B^3 - D^3$. Let $B - A$ be the root of the first one that is to be found, and let $B^2A/D^2 - D$ be the root of the second. Forming the cubes and comparing them with $B^3 - D^3$, it will be found that $3D^3B/(B^3 + D^3)$ equals A. The root of the first cube to be found, therefore, is $[B(B^3 - 2D^3)]/(B^3 + D^3)$ and of the second is $[D(2B^3 - D^3)]/(B^3 + D^3)$. And the sum of the two cubes of these is equal to $B^3 - D^3$."

That is,

$$B^3 - D^3 = \left(\frac{B(B^3 - 2D^3)}{B^3 + D^3}\right)^3 + \left(\frac{D(2B^3 - D^3)}{B^3 + D^3}\right)^3.$$

With so many grad students here, there has to be pizza

By setting $B = x$ and $D = -y$, Viète's formula becomes (2):

$$x^3 + y^3 = \left(\frac{x(x^3 + 2y^3)}{x^3 - y^3}\right)^3 + \left(\frac{y(y^3 + 2x^3)}{y^3 - x^3}\right)^3.$$

With so many grad students here, there has to be pizza

By setting $B = x$ and $D = -y$, Viète's formula becomes (2):

$$x^3 + y^3 = \left(\frac{x(x^3 + 2y^3)}{x^3 - y^3} \right)^3 + \left(\frac{y(y^3 + 2x^3)}{y^3 - x^3} \right)^3 .$$

Jeremy Rouse (who was a postdoc here last year) and I have just written a paper in which we examine the more general equation

$$x^3 + y^3 = p^3(x, y) + q^3(x, y) \tag{6}$$

for homogeneous rational functions $p, q \in \mathbb{C}(x, y)$. You can find it on the arXiv.

Be a pal, just slip a pepperoni into my disk drive

To examine (6), we take a common denominator for p, q and rewrite as:

$$x^3 + y^3 = \left(\frac{f(x,y)}{h(x,y)}\right)^3 + \left(\frac{g(x,y)}{h(x,y)}\right)^3$$

Be a pal, just slip a pepperoni into my disk drive

To examine (6), we take a common denominator for p, q and rewrite as:

$$\begin{aligned} x^3 + y^3 &= \left(\frac{f(x,y)}{h(x,y)}\right)^3 + \left(\frac{g(x,y)}{h(x,y)}\right)^3 \qquad (7) \\ \implies h(x,y)^3(x^3+y^3) &= f(x,y)^3 + g(x,y)^3. \end{aligned}$$

Be a pal, just slip a pepperoni into my disk drive

To examine (6), we take a common denominator for p, q and rewrite as:

$$\begin{aligned} x^3 + y^3 &= \left(\frac{f(x,y)}{h(x,y)}\right)^3 + \left(\frac{g(x,y)}{h(x,y)}\right)^3 \\ \implies h(x,y)^3(x^3+y^3) &= f(x,y)^3 + g(x,y)^3. \end{aligned} \tag{7}$$

It follows that if $\pi(x,y)$ is irreducible and π divides any two of $\{f,g,h\}$, then it divides the third. Also note that f and g may be permuted and cube roots of unity ω^j may appear.
Assume that f, g, h are forms (that is, homogeneous). If $\deg f = \deg g = d$, then we call (7) a *solution of degree d*.

Be a pal, just slip a pepperoni into my disk drive

To examine (6), we take a common denominator for p, q and rewrite as:

$$x^3 + y^3 = \left(\frac{f(x,y)}{h(x,y)}\right)^3 + \left(\frac{g(x,y)}{h(x,y)}\right)^3 \tag{7}$$
$$\implies h(x,y)^3(x^3+y^3) = f(x,y)^3 + g(x,y)^3.$$

It follows that if $\pi(x,y)$ is irreducible and π divides any two of $\{f, g, h\}$, then it divides the third. Also note that f and g may be permuted and cube roots of unity ω^j may appear.
Assume that f, g, h are forms (that is, homogeneous). If $\deg f = \deg g = d$, then we call (7) a *solution of degree* d.

Maybe there's a reason (8) isn't in the literature

There's an obvious solution of degree 1: $(f, g, h) = (x, y, 1)$.
Viète's solution has degree 4, but there's also one of degree 3.

Maybe there's a reason (8) isn't in the literature

There's an obvious solution of degree 1: $(f, g, h) = (x, y, 1)$. Viète's solution has degree 4, but there's also one of degree 3. Let $\zeta = \zeta_{12} = \frac{\sqrt{3}}{2} + \frac{i}{2}$, and observe that $\zeta + \zeta^{-1} = \sqrt{3}$ and $\zeta^3 + \zeta^{-3} = i - i = 0$.

Maybe there's a reason (8) isn't in the literature

There's an obvious solution of degree 1: $(f, g, h) = (x, y, 1)$. Viète's solution has degree 4, but there's also one of degree 3. Let $\zeta = \zeta_{12} = \frac{\sqrt{3}}{2} + \frac{i}{2}$, and observe that $\zeta + \zeta^{-1} = \sqrt{3}$ and $\zeta^3 + \zeta^{-3} = i - i = 0$. Then

$$\begin{aligned}&(\zeta u + \zeta^{-1} v)^3 + (\zeta^{-1} u + \zeta v)^3 \\ &= (\zeta^3 + \zeta^{-3})(u^3 + v^3) + 3(\zeta + \zeta^{-1})(u^2 v + u v^2)\end{aligned}$$

Maybe there's a reason (8) isn't in the literature

There's an obvious solution of degree 1: $(f, g, h) = (x, y, 1)$. Viète's solution has degree 4, but there's also one of degree 3. Let $\zeta = \zeta_{12} = \frac{\sqrt{3}}{2} + \frac{i}{2}$, and observe that $\zeta + \zeta^{-1} = \sqrt{3}$ and $\zeta^3 + \zeta^{-3} = i - i = 0$. Then

$$\begin{aligned} &(\zeta u + \zeta^{-1} v)^3 + (\zeta^{-1} u + \zeta v)^3 \\ &= (\zeta^3 + \zeta^{-3})(u^3 + v^3) + 3(\zeta + \zeta^{-1})(u^2 v + u v^2) \\ &= 3\sqrt{3} uv(u + v). \end{aligned}$$

Maybe there's a reason (8) isn't in the literature

There's an obvious solution of degree 1: $(f, g, h) = (x, y, 1)$. Viète's solution has degree 4, but there's also one of degree 3. Let $\zeta = \zeta_{12} = \frac{\sqrt{3}}{2} + \frac{i}{2}$, and observe that $\zeta + \zeta^{-1} = \sqrt{3}$ and $\zeta^3 + \zeta^{-3} = i - i = 0$. Then

$$\begin{aligned} &(\zeta u + \zeta^{-1} v)^3 + (\zeta^{-1} u + \zeta v)^3 \\ &= (\zeta^3 + \zeta^{-3})(u^3 + v^3) + 3(\zeta + \zeta^{-1})(u^2 v + u v^2) \\ &= 3\sqrt{3} uv(u + v). \end{aligned}$$

After $(u, v) \mapsto (x^3, y^3)$, this is equivalent to:

$$x^3 + y^3 = \left(\frac{\zeta x^3 + \zeta^{-1} y^3}{\sqrt{3} xy} \right)^3 + \left(\frac{\zeta^{-1} x^3 + \zeta y^3}{\sqrt{3} xy} \right)^3. \qquad (8)$$

Maybe there's a reason (8) isn't in the literature

There's an obvious solution of degree 1: $(f, g, h) = (x, y, 1)$. Viète's solution has degree 4, but there's also one of degree 3. Let $\zeta = \zeta_{12} = \frac{\sqrt{3}}{2} + \frac{i}{2}$, and observe that $\zeta + \zeta^{-1} = \sqrt{3}$ and $\zeta^3 + \zeta^{-3} = i - i = 0$. Then

$$\begin{aligned} &(\zeta u + \zeta^{-1} v)^3 + (\zeta^{-1} u + \zeta v)^3 \\ &= (\zeta^3 + \zeta^{-3})(u^3 + v^3) + 3(\zeta + \zeta^{-1})(u^2 v + uv^2) \\ &= 3\sqrt{3} uv(u + v). \end{aligned}$$

After $(u, v) \mapsto (x^3, y^3)$, this is equivalent to:

$$x^3 + y^3 = \left(\frac{\zeta x^3 + \zeta^{-1} y^3}{\sqrt{3} xy}\right)^3 + \left(\frac{\zeta^{-1} x^3 + \zeta y^3}{\sqrt{3} xy}\right)^3. \qquad (8)$$

Let's call this the *small* solution.

Maybe there's a reason (8) isn't in the literature

There's an obvious solution of degree 1: $(f, g, h) = (x, y, 1)$.
Viète's solution has degree 4, but there's also one of degree 3.
Let $\zeta = \zeta_{12} = \frac{\sqrt{3}}{2} + \frac{i}{2}$, and observe that $\zeta + \zeta^{-1} = \sqrt{3}$ and $\zeta^3 + \zeta^{-3} = i - i = 0$. Then

$$\begin{aligned}(\zeta u + \zeta^{-1} v)^3 &+ (\zeta^{-1} u + \zeta v)^3 \\ = (\zeta^3 + \zeta^{-3})(u^3 + v^3) &+ 3(\zeta + \zeta^{-1})(u^2 v + uv^2) \\ &= 3\sqrt{3}uv(u + v).\end{aligned}$$

After $(u, v) \mapsto (x^3, y^3)$, this is equivalent to:

$$x^3 + y^3 = \left(\frac{\zeta x^3 + \zeta^{-1} y^3}{\sqrt{3}xy}\right)^3 + \left(\frac{\zeta^{-1} x^3 + \zeta y^3}{\sqrt{3}xy}\right)^3. \qquad (8)$$

Let's call this the *small* solution.
If someone has seen (8) in the literature, please let me know!

There are no small solutions, only small speakers

There are two solutions of degree 7 which are complex conjugates of each other. Here's one of them.

$$f(x,y) = x(x^6 + (-1+3\sqrt{3}i)(x^3y^3 + y^6)),$$
$$g(x,y) = y((-1+3\sqrt{3}i)(x^6 + x^3y^3) + y^6),$$
$$h(x,y) = x^6 + \left(\frac{5-3\sqrt{3}i}{2}\right)x^3y^3 + y^6.$$

There are no small solutions, only small speakers

There are two solutions of degree 7 which are complex conjugates of each other. Here's one of them.

$$
\begin{aligned}
f(x,y) &= x(x^6 + (-1 + 3\sqrt{3}i)(x^3y^3 + y^6)),\\
g(x,y) &= y((-1 + 3\sqrt{3}i)(x^6 + x^3y^3) + y^6),\\
h(x,y) &= x^6 + \left(\frac{5 - 3\sqrt{3}i}{2}\right) x^3y^3 + y^6.
\end{aligned}
$$

There is one degree 9 solution, with real integral coefficients:

$$
\begin{aligned}
f(x,y) &= x^9 + 6x^6y^3 + 3x^3y^6 - y^9,\\
g(x,y) &= -x^9 + 3x^6y^3 + 6x^3y^6 + y^9,\\
h(x,y) &= 3xy(x^6 + x^3y^3 + y^6).
\end{aligned}
$$

Put your tray tables in the upright and locked position

In addition to the symmetries mentioned earlier, there is a natural composition of two solutions $(p_1, q_1) \circ (p_2, q_2)$:

$$\begin{gathered} x^3 + y^3 = p_1^3(x,y) + q_1^3(x,y) = p_2^3(x,y) + q_2^3(x,y) \\ \implies p_1^3(p_2(x,y), q_2(x,y)) + q_1^3(p_2(x,y), q_2(x,y)) \\ = p_2^3(x,y) + q_2^3(x,y) = x^3 + y^3. \end{gathered}$$

Put your tray tables in the upright and locked position

In addition to the symmetries mentioned earlier, there is a natural composition of two solutions $(p_1, q_1) \circ (p_2, q_2)$:

$$\begin{aligned} x^3 + y^3 = p_1^3(x,y) + q_1^3(x,y) = p_2^3(x,y) + q_2^3(x,y) \\ \implies p_1^3(p_2(x,y), q_2(x,y)) + q_1^3(p_2(x,y), q_2(x,y)) \\ = p_2^3(x,y) + q_2^3(x,y) = x^3 + y^3. \end{aligned}$$

The small solution composed with itself gives the (real) degree 9 solution: the roots of unity cancel! Viète's solution and the small solution commute, giving the (unique) solution of degree 12.

Put your tray tables in the upright and locked position

In addition to the symmetries mentioned earlier, there is a natural composition of two solutions $(p_1, q_1) \circ (p_2, q_2)$:

$$\begin{gathered} x^3 + y^3 = p_1^3(x,y) + q_1^3(x,y) = p_2^3(x,y) + q_2^3(x,y) \\ \implies p_1^3(p_2(x,y), q_2(x,y)) + q_1^3(p_2(x,y), q_2(x,y)) \\ = p_2^3(x,y) + q_2^3(x,y) = x^3 + y^3. \end{gathered}$$

The small solution composed with itself gives the (real) degree 9 solution: the roots of unity cancel! Viète's solution and the small solution commute, giving the (unique) solution of degree 12.

One of our tools is a theorem which might have been known in the 19th century literature, but I've never found it.

Maybe there's a reason it isn't in the literature

Theorem

Suppose $p \in \mathbb{C}[x_1, \dots, x_n]$*. Then there exist* $f, g \in \mathbb{C}[x_1, \dots, x_n]$ *such that* $p = f^3 + g^3$ *if and only if* p *is a cube, or* $p = q_1 q_2 q_3$*, where* q_i*'s are linearly dependent, but pairwise non-proportional.*

Maybe there's a reason it isn't in the literature

Theorem

Suppose $p \in \mathbb{C}[x_1, \dots, x_n]$. *Then there exist* $f, g \in \mathbb{C}[x_1, \dots, x_n]$ *such that* $p = f^3 + g^3$ *if and only if* p *is a cube, or* $p = q_1 q_2 q_3$, *where* q_i*'s are linearly dependent, but pairwise non-proportional.*

Proof.

Assume p is not a cube. Then $p = (f + g)(f + \omega g)(f + \omega^2 g)$ is such a factorization.

Maybe there's a reason it isn't in the literature

Theorem

Suppose $p \in \mathbb{C}[x_1, \dots, x_n]$*. Then there exist* $f, g \in \mathbb{C}[x_1, \dots, x_n]$ *such that* $p = f^3 + g^3$ *if and only if* p *is a cube, or* $p = q_1 q_2 q_3$*, where* q_i*'s are linearly dependent, but pairwise non-proportional.*

Proof.

Assume p is not a cube. Then $p = (f+g)(f+\omega g)(f+\omega^2 g)$ is such a factorization.

Maybe there's a reason it isn't in the literature

Theorem

Suppose $p \in \mathbb{C}[x_1, \dots, x_n]$*. Then there exist* $f, g \in \mathbb{C}[x_1, \dots, x_n]$ *such that* $p = f^3 + g^3$ *if and only if* p *is a cube, or* $p = q_1 q_2 q_3$*, where* q_i*'s are linearly dependent, but pairwise non-proportional.*

Proof.

Assume p is not a cube. Then $p = (f+g)(f+\omega g)(f+\omega^2 g)$ is such a factorization.
If $p = q_1 q_2 q_3$ and $q_3 = aq_1 + bq_2$ with $ab \neq 0$, then

$$\left(\frac{\zeta\ a\ q_1 + \zeta^{-1}\ b\ q_2}{\sqrt{3}(ab)^{1/3}}\right)^3 + \left(\frac{\zeta^{-1}\ a\ q_1 + \zeta\ b\ q_2}{\sqrt{3}(ab)^{1/3}}\right)^3$$

$$= q_1 q_2 (a\ q_1 + b\ q_2) = p.$$

□

What's “even” and “odd” in ternary?

This theorem can used to analyze $(x^3 + y^3)h(x, y)^3$.
For example, $x^3, y^3, (x^3 + y^3)$ are linearly dependent, hence $x^3y^3(x^3 + y^3)$ is a sum of two cubes. This is the small solution.

What's "even" and "odd" in ternary?

This theorem can used to analyze $(x^3 + y^3)h(x, y)^3$.
For example, $x^3, y^3, (x^3 + y^3)$ are linearly dependent, hence $x^3y^3(x^3 + y^3)$ is a sum of two cubes. This is the small solution.
Less trivially, looking at the exponents mod 3, we see that

$$(x + y)(x - y)^3 = (x^4 + 2xy^3) - (2x^3y + y^4)$$
$$(x + \omega y)(x - \omega y)^3 = (x^4 + 2xy^3) - \omega(2x^3y + y^4)$$
$$(x + \omega^2 y)(x - \omega^2 y)^3 = (x^4 + 2xy^3) - \omega^2(2x^3y + y^4)$$

What's "even" and "odd" in ternary?

This theorem can used to analyze $(x^3 + y^3)h(x,y)^3$.
For example, $x^3, y^3, (x^3 + y^3)$ are linearly dependent, hence $x^3y^3(x^3 + y^3)$ is a sum of two cubes. This is the small solution.
Less trivially, looking at the exponents mod 3, we see that

$$(x+y)(x-y)^3 = (x^4 + 2xy^3) - (2x^3y + y^4)$$
$$(x+\omega y)(x-\omega y)^3 = (x^4 + 2xy^3) - \omega(2x^3y + y^4)$$
$$(x+\omega^2 y)(x-\omega^2 y)^3 = (x^4 + 2xy^3) - \omega^2(2x^3y + y^4)$$

are linearly dependent, hence their product,

$$\begin{gathered}(x+y)(x+\omega y)(x+\omega^2 y)(x-y)^3(x-\omega y)^3(x-\omega^2 y)^3 \\ = (x^3+y^3)(x^3-y^3)^3,\end{gathered}$$

is a sum of two cubes. If you work out the details, you recover (2).

Oh my goodness, Brightshirt enters the 20th century!

The second tool is familiar in elliptic curves. The line through two points on the curve $X^3 + Y^3 = A$ intersects the curve in a third point, which, after reflection, is called the *sum* of the two points. This defines an abelian group.

Oh my goodness, Brightshirt enters the 20th century!

The second tool is familiar in elliptic curves. The line through two points on the curve $X^3 + Y^3 = A$ intersects the curve in a third point, which, after reflection, is called the *sum* of the two points. This defines an abelian group.

Assuming $X_j^3 + Y_j^3 = A$, the addition law works out to be

$$(X_1, Y_1) + (X_2, Y_2) = (X_3, Y_3),$$

where

$$X_3 = \frac{A(Y_1 - Y_2) + X_1X_2(X_1Y_2 - X_2Y_1)}{(X_1^2X_2 + Y_1^2Y_2) - (X_1X_2^2 + Y_1Y_2^2)},$$
$$Y_3 = \frac{A(X_1 - X_2) + Y_1Y_2(X_2Y_1 - X_1Y_2)}{(X_1^2X_2 + Y_1^2Y_2) - (X_1X_2^2 + Y_1Y_2^2)}.$$

Jeremy really proved all the hard stuff here

This formula breaks down when the two points coincide; instead, take a line tangent to the curve at (X_1, Y_1). By implicit differentiation, the slope is $-\frac{X_1^2}{Y_1^2}$ and we seek t so that

$$(X_1 - t)^3 + \left(Y_1 + t \cdot \frac{X_1^2}{Y_1^2}\right)^3 = X_1^3 + Y_1^3$$

It turns out that there is a double root at $t = 0$ and a single root at $t = -\frac{3X_1 Y_1^3}{X_1^3 - Y_1^3}$. Putting this value of t above gives equation (2).

Jeremy really proved all the hard stuff here

This formula breaks down when the two points coincide; instead, take a line tangent to the curve at (X_1, Y_1). By implicit differentiation, the slope is $-\frac{X_1^2}{Y_1^2}$ and we seek t so that

$$(X_1 - t)^3 + \left(Y_1 + t \cdot \frac{X_1^2}{Y_1^2}\right)^3 = X_1^3 + Y_1^3$$

It turns out that there is a double root at $t = 0$ and a single root at $t = -\frac{3X_1Y_1^3}{X_1^3 - Y_1^3}$. Putting this value of t above gives equation (2). Believe it or not, this is, formally, what Viète was doing! I doubt he knew about elliptic curves, but he was one of the first people to study cubics and he must have known that this particular substitution would lead to a nice solution.

Don't worry, this won't be on the test

Elliptic curves are usually discussed where A is a number or element of a finite field. Let's suppose $X, Y, A \in \mathbb{C}(t)$, and $A = 1 + t^3$. Then the equation is

$$X^3(t) + Y^3(t) = 1 + t^3 \tag{9}$$

and if we homogenize (9), by setting $t = y/x$ and multiplying both sides by x^3, then we get (6). In order to fit in this interpretation, though, we recognize that every solution (p, q) corresponds to 18 points on the curve: $(\omega^j p, \omega^k q)$ and $(\omega^j q, \omega^k p)$, $0 \le j, k \le 2$.

Don't worry, this won't be on the test

Elliptic curves are usually discussed where A is a number or element of a finite field. Let's suppose $X, Y, A \in \mathbb{C}(t)$, and $A = 1 + t^3$. Then the equation is

$$X^3(t) + Y^3(t) = 1 + t^3 \tag{9}$$

and if we homogenize (9), by setting $t = y/x$ and multiplying both sides by x^3, then we get (6). In order to fit in this interpretation, though, we recognize that every solution (p, q) corresponds to 18 points on the curve: $(\omega^j p, \omega^k q)$ and $(\omega^j q, \omega^k p)$, $0 \le j, k \le 2$.

The famous Mordell-Weil Theorem says that the group of rational points on an elliptic curve is finitely generated, and it also applies to curves over $\mathbb{C}(t)$ such as this. Under the definition given above, Viète's solution turns out to be $-2(x, y)$ and the small solution is $(\omega x, y) + (\omega^2 y, x)$.

Didn't say anything about the comps

We now recall our notation and give some results. Suppose

$$x^3 + y^3 = p^3(x,y) + q^3(x,y) = \left(\frac{f(x,y)}{h(x,y)}\right)^3 + \left(\frac{g(x,y)}{h(x,y)}\right)^3$$

and the solution has degree d.

Didn't say anything about the comps

We now recall our notation and give some results. Suppose

$$x^3 + y^3 = p^3(x,y) + q^3(x,y) = \left(\frac{f(x,y)}{h(x,y)}\right)^3 + \left(\frac{g(x,y)}{h(x,y)}\right)^3$$

and the solution has degree d.

- $q(x,y) = p(y,x)$ (up to powers of ω).

Didn't say anything about the comps

We now recall our notation and give some results. Suppose

$$x^3 + y^3 = p^3(x,y) + q^3(x,y) = \left(\frac{f(x,y)}{h(x,y)}\right)^3 + \left(\frac{g(x,y)}{h(x,y)}\right)^3$$

and the solution has degree d.

- $q(x,y) = p(y,x)$ (up to powers of ω).
- $p, q \in \mathbb{Q}(\omega)(x,y)$.

Didn't say anything about the comps

We now recall our notation and give some results. Suppose

$$x^3 + y^3 = p^3(x,y) + q^3(x,y) = \left(\frac{f(x,y)}{h(x,y)}\right)^3 + \left(\frac{g(x,y)}{h(x,y)}\right)^3$$

and the solution has degree d.

- $q(x,y) = p(y,x)$ (up to powers of ω).
- $p, q \in \mathbb{Q}(\omega)(x,y)$.
- There is a solution in $\mathbb{Q}(x,y)$ iff d is a square.

Didn't say anything about the comps

We now recall our notation and give some results. Suppose

$$x^3 + y^3 = p^3(x,y) + q^3(x,y) = \left(\frac{f(x,y)}{h(x,y)}\right)^3 + \left(\frac{g(x,y)}{h(x,y)}\right)^3$$

and the solution has degree d.

- $q(x,y) = p(y,x)$ (up to powers of ω).
- $p, q \in \mathbb{Q}(\omega)(x,y)$.
- There is a solution in $\mathbb{Q}(x,y)$ iff d is a square.
- Any two solutions commute under composition, up to multiplication by cube roots of unity.

Didn't say anything about the comps

We now recall our notation and give some results. Suppose

$$x^3 + y^3 = p^3(x, y) + q^3(x, y) = \left(\frac{f(x, y)}{h(x, y)}\right)^3 + \left(\frac{g(x, y)}{h(x, y)}\right)^3$$

and the solution has degree d.

- $q(x, y) = p(y, x)$ (up to powers of ω).
- $p, q \in \mathbb{Q}(\omega)(x, y)$.
- There is a solution in $\mathbb{Q}(x, y)$ iff d is a square.
- Any two solutions commute under composition, up to multiplication by cube roots of unity.
- Any solution of degree $3k$ is the composition of the small solution with a solution of degree k.

Didn't say anything about the comps

We now recall our notation and give some results. Suppose

$$x^3 + y^3 = p^3(x,y) + q^3(x,y) = \left(\frac{f(x,y)}{h(x,y)}\right)^3 + \left(\frac{g(x,y)}{h(x,y)}\right)^3$$

and the solution has degree d.

- $q(x,y) = p(y,x)$ (up to powers of ω).
- $p, q \in \mathbb{Q}(\omega)(x,y)$.
- There is a solution in $\mathbb{Q}(x,y)$ iff d is a square.
- Any two solutions commute under composition, up to multiplication by cube roots of unity.
- Any solution of degree $3k$ is the composition of the small solution with a solution of degree k.
- No monomial occuring in any f, g, h has an exponent $\equiv 2$ mod 3.

It's not just a good idea, it's the law

- The set of solutions form the group $\mathbb{Z} + \mathbb{Z} + \mathbb{Z}_3$, with generators (x, y), $(\omega x, \omega y)$ and torsion involves ω^j. The solution $m(x, y) + n(\omega x, \omega y)$ has degree $m^2 - mn + n^2$.

It's not just a good idea, it's the law

- The set of solutions form the group $\mathbb{Z} + \mathbb{Z} + \mathbb{Z}_3$, with generators (x, y), $(\omega x, \omega y)$ and torsion involves ω^j. The solution $m(x, y) + n(\omega x, \omega y)$ has degree $m^2 - mn + n^2$.
- The subgroup $\mathbb{Z} + \mathbb{Z}$ is actually **ring**-homomorphic to $\mathbb{Z}[\omega]$, under the operations of point addition and composition.

It's not just a good idea, it's the law

- The set of solutions form the group $\mathbb{Z}+\mathbb{Z}+\mathbb{Z}_3$, with generators (x,y), $(\omega x,\omega y)$ and torsion involves ω^j. The solution $m(x,y)+n(\omega x,\omega y)$ has degree m^2-mn+n^2.
- The subgroup $\mathbb{Z}+\mathbb{Z}$ is actually **ring**-homomorphic to $\mathbb{Z}[\omega]$, under the operations of point addition and composition.
- Let $a(d)$ denote the number of solutions of degree d, then

$$1+6\sum_{d=1}^{\infty}a(d)x^d=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}x^{m^2-mn+n^2}\implies$$

$$\sum_{d=1}^{\infty}a(d)z^d=\sum_{i=0}^{\infty}\left(\frac{x^{3i+1}}{1-x^{3i+1}}-\frac{x^{3i+2}}{1-x^{3i+2}}\right).$$

It’s not just a good idea, it’s the law

- The set of solutions form the group $\mathbb{Z}+\mathbb{Z}+\mathbb{Z}_3$, with generators (x,y), $(\omega x, \omega y)$ and torsion involves ω^j. The solution $m(x,y)+n(\omega x,\omega y)$ has degree m^2-mn+n^2.
- The subgroup $\mathbb{Z}+\mathbb{Z}$ is actually **ring**-homomorphic to $\mathbb{Z}[\omega]$, under the operations of point addition and composition.
- Let $a(d)$ denote the number of solutions of degree d, then

$$1+6\sum_{d=1}^{\infty} a(d)x^d = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} x^{m^2-mn+n^2} \implies$$

$$\sum_{d=1}^{\infty} a(d)z^d = \sum_{i=0}^{\infty}\left(\frac{x^{3i+1}}{1-x^{3i+1}} - \frac{x^{3i+2}}{1-x^{3i+2}}\right).$$

- The number of solutions of degree d is the number of factors of d congruent to 1 mod 3 minus the number congruent to 2 mod 3. Any degree has the form $m^2\prod_j p_j$, $p_j \equiv 1 \bmod 3$.

You'll not see nothing like the mighty quintic

Recall (3), proved by Desboves (1880) and Elkies (1995): let

$$f_1(x,y) = x^2 + \sqrt{2}\ x\ y - y^2, f_2(x,y) = i\ x^2 - \sqrt{2}\ x\ y + i\ y^2$$
$$f_3(x,y) = -x^2 + \sqrt{2}\ x\ y + y^2, f_4(x,y) = -i\ x^2 - \sqrt{2}\ x\ y - i\ y^2$$

Then $\sum_{i=1}^4 f_i^5 = 0$.

You'll not see nothing like the mighty quintic

Recall (3), proved by Desboves (1880) and Elkies (1995): let

$$f_1(x,y) = x^2 + \sqrt{2}\ x\ y - y^2, f_2(x,y) = i\ x^2 - \sqrt{2}\ x\ y + i\ y^2$$
$$f_3(x,y) = -x^2 + \sqrt{2}\ x\ y + y^2, f_4(x,y) = -i\ x^2 - \sqrt{2}\ x\ y - i\ y^2$$

Then $\sum_{i=1}^{4} f_i^5 = 0$. This was derived by taking the sum

$$\sum_{k=0}^{3} (i^k x^2 + i^{2k} a\ x\ y + i^{3k} y^2)^5 = 40a(a^2+2)(x^7y^3 + x^3y^7),$$

and setting first $a = \sqrt{-2}$ and then $y \mapsto iy$.

You'll not see nothing like the mighty quintic

Recall (3), proved by Desboves (1880) and Elkies (1995): let

$$f_1(x,y) = x^2 + \sqrt{2}\ x\ y - y^2, f_2(x,y) = i\ x^2 - \sqrt{2}\ x\ y + i\ y^2$$
$$f_3(x,y) = -x^2 + \sqrt{2}\ x\ y + y^2, f_4(x,y) = -i\ x^2 - \sqrt{2}\ x\ y - i\ y^2$$

Then $\sum_{i=1}^4 f_i^5 = 0$. This was derived by taking the sum

$$\sum_{k=0}^{3}(i^k x^2 + i^{2k} a\ x\ y + i^{3k} y^2)^5 = 40a(a^2+2)(x^7y^3 + x^3y^7),$$

and setting first $a = \sqrt{-2}$ and then $y \mapsto iy$.
The interplay of the roots of unity makes it unsurprising that

$$\sum_{i=1}^{4} f_i = \sum_{i=1}^{4} f_i^2 = 0$$

as well. This is actually, however, too much of a good thing.

True story: “sextic” is spam-worthy for CITES

Note that the equations $\sum f_i = \sum f_i^2 = 0$ define the intersection of a plane and a sphere in $\mathbb{C}^4$. This is, projectively, a curve. Unless something special is going on, this curve shouldn't contain another curve (f_1, f_2, f_3, f_4).

True story: “sextic” is spam-worthy for CITES

Note that the equations $\sum f_i = \sum f_i^2 = 0$ define the intersection of a plane and a sphere in $\mathbb{C}^4$. This is, projectively, a curve. Unless something special is going on, this curve shouldn't contain another curve (f_1, f_2, f_3, f_4).
What's special is that the ideal generated by $\sum_{i=1}^4 x_i$ and $\sum_{i=1}^4 x_i^2$ contains $\sum_{i=1}^4 x_i^5$. Proof in a bit.

True story: “sextic” is spam-worthy for CITES

Note that the equations $\sum f_i = \sum f_i^2 = 0$ define the intersection of a plane and a sphere in $\mathbb{C}^4$. This is, projectively, a curve. Unless something special is going on, this curve shouldn't contain another curve (f_1, f_2, f_3, f_4).

What's special is that the ideal generated by $\sum_{i=1}^4 x_i$ and $\sum_{i=1}^4 x_i^2$ contains $\sum_{i=1}^4 x_i^5$. Proof in a bit.

If $f_4 = -(f_1 + f_2 + f_3)$, then the sum of squares becomes essentially a Pythagorean triple, which we know how to parameterize:

$$f_1^2 + f_2^2 + f_3^2 + (f_1 + f_2 + f_3)^2 = 0 \implies$$

True story: "sextic" is spam-worthy for CITES

Note that the equations $\sum f_i = \sum f_i^2 = 0$ define the intersection of a plane and a sphere in $\mathbb{C}^4$. This is, projectively, a curve. Unless something special is going on, this curve shouldn't contain another curve (f_1, f_2, f_3, f_4).

What's special is that the ideal generated by $\sum_{i=1}^4 x_i$ and $\sum_{i=1}^4 x_i^2$ contains $\sum_{i=1}^4 x_i^5$. Proof in a bit.

If $f_4 = -(f_1 + f_2 + f_3)$, then the sum of squares becomes essentially a Pythagorean triple, which we know how to parameterize:

$$f_1^2 + f_2^2 + f_3^2 + (f_1 + f_2 + f_3)^2 = 0 \implies$$

$$(f_1 - f_3)^2 + 2(f_1 + f_3)^2 = -(f_1 + 2f_2 + f_3)^2 \quad \text{``} \implies \text{''}$$

True story: "sextic" is spam-worthy for CITES

Note that the equations $\sum f_i = \sum f_i^2 = 0$ define the intersection of a plane and a sphere in $\mathbb{C}^4$. This is, projectively, a curve. Unless something special is going on, this curve shouldn't contain another curve (f_1, f_2, f_3, f_4).

What's special is that the ideal generated by $\sum_{i=1}^4 x_i$ and $\sum_{i=1}^4 x_i^2$ contains $\sum_{i=1}^4 x_i^5$. Proof in a bit.

If $f_4 = -(f_1 + f_2 + f_3)$, then the sum of squares becomes essentially a Pythagorean triple, which we know how to parameterize:

$$f_1^2 + f_2^2 + f_3^2 + (f_1 + f_2 + f_3)^2 = 0 \implies$$

$$(f_1 - f_3)^2 + 2(f_1 + f_3)^2 = -(f_1 + 2f_2 + f_3)^2 \quad \text{``} \implies \text{''}$$

$$f_1 - f_3 = x^2 - y^2, \sqrt{2}(f_1 + f_3) = 2xy, -i(f_1 + 2f_2 + f_3) = x^2 + y^2$$

We can solve for the f_i's to recover the Desboves-Elkies example.

We need the Frobenius problem.

We need the Frobenius problem.
Let m and n be relatively prime positive integers > 1 and let $A(m, n)$ denote the set of positive integers which **cannot** be written as $am + bn$ for non-negative integers (a, b). Sylvester showed in 1884 that $A(m, n)$ has $(m-1)(n-1)/2$ elements (m and n can't both be even), the largest of which is $mn - m - n$.

We need the Frobenius problem.

Let m and n be relatively prime positive integers > 1 and let $A(m,n)$ denote the set of positive integers which **cannot** be written as $am + bn$ for non-negative integers (a,b). Sylvester showed in 1884 that $A(m,n)$ has $(m-1)(n-1)/2$ elements (m and n can't both be even), the largest of which is $mn - m - n$.

Define the k-th power sum function:

$$M_{n,k}(x_1,\ldots,x_n) = \sum_{j=1}^{n} x_j^k.$$

We need the Frobenius problem.
Let m and n be relatively prime positive integers > 1 and let $A(m,n)$ denote the set of positive integers which **cannot** be written as $am + bn$ for non-negative integers (a,b). Sylvester showed in 1884 that $A(m,n)$ has $(m-1)(n-1)/2$ elements (m and n can't both be even), the largest of which is $mn - m - n$.

Define the k-th power sum function:

$$M_{n,k}(x_1,\dots,x_n) = \sum_{j=1}^{n} x_j^k.$$

It is not hard to show that

$$\begin{gathered} M_{n,1}(x) = M_{n,2}(x) = \cdots = M_{n,r}(x) = 0 \\ \iff e_{n,1}(x) = e_{n,2}(x) = \cdots = e_{n,r}(x) = 0 \end{gathered}$$

where $e_{n,k}(x)$ is the k-th elementary symmetric function in $(x_1,\dots,x_n)$. Equivalently, the two sets of symmetric functions determine the same ideal.

Theorem

Suppose $x \in \mathbb{C}^n$ *is such that* $M_{n,r}(x) = 0$ *for* $r = 1, \dots, n-2$. *If* $N \in A(n-1, n)$*, then* $M_{n,N}(x) = 0$ *as well. Alternatively, if* N *is not expressible as* $a(n-1) + bn$*, then*

$$\sum_{j=1}^{n} x_j^N \in \left(\sum_{j=1}^{n} x_j, \sum_{j=1}^{n} x_j^2, \dots, \sum_{j=1}^{n} x_j^{n-2} \right).$$

Note that if $n = 4$, then $A(3,4) = \{1, 2, 5\}$. This completes the derivation of (3). The largest element in $A(n-1, n)$ is $n^2 - 3n + 1$.

Proof.

The hypothesis implies that $e_{n,r}(x) = 0$ for $r = 1, \dots, n-2$ as well, where the $e_{n,r}$'s are the elementary symmetric functions.

Proof.

The hypothesis implies that $e_{n,r}(x) = 0$ for $r = 1, \ldots, n-2$ as well, where the $e_{n,r}$'s are the elementary symmetric functions.

Proof.

The hypothesis implies that $e_{n,r}(x) = 0$ for $r = 1, \ldots, n-2$ as well, where the $e_{n,r}$'s are the elementary symmetric functions.

Suppose P is *any* symmetric form of degree N, not necessarily the sum of the N-th powers. By Newton's theorem on symmetric polynomials, P can be expressed in the form

$$\sum_{i_1+2i_2+\cdots+ni_n=N} c_{i_1,\ldots,i_n} e_{n,1}^{i_1} \ldots e_{n,n}^{i_n}$$

Proof.

The hypothesis implies that $e_{n,r}(x) = 0$ for $r = 1, \dots, n-2$ as well, where the $e_{n,r}$'s are the elementary symmetric functions.

Suppose P is *any* symmetric form of degree N, not necessarily the sum of the N-th powers. By Newton's theorem on symmetric polynomials, P can be expressed in the form

$$\sum_{i_1+2i_2+\cdots+ni_n=N} c_{i_1,\dots,i_n} e_{n,1}^{i_1} \dots e_{n,n}^{i_n}$$

But N can't be written as $\sum k i_k$ without having $i_k > 0$ for some $k \le n-2$, so each summand contains some $e_{n,k}$ with $k \le n-2$. Thus, P is in the ideal $(e_{n,1}, \dots, e_{n,n-2})$, which is equal to the ideal $(M_{n,1}, \dots, M_{n,n-2})$. $\square$

Randy, I use the word "genus" here. Happy?

For $n \geq 5$, the intersection $\cap_{r=1}^{n-2} M_{n,r}$ has positive genus and so has no polynomial parameterization: despite the Theorem, there are no versions of Desboves-Elkies in higher degrees.

Randy, I use the word “genus” here. Happy?

For $n \geq 5$, the intersection $\cap_{r=1}^{n-2} M_{n,r}$ has positive genus and so has no polynomial parameterization: despite the Theorem, there are no versions of Desboves-Elkies in higher degrees.

I spent some time searching for other “interesting” identities of this kind and found this one:

$$\sum_{k=0}^{4} (\zeta_5^k x^2 + a\ x\ y + \zeta_5^{-k})^{14} =$$

$$f(a)(x^{24}y^4 + x^4y^{24}) + g(a)(x^{19}y^9 + x^9y^{19}) + h(a)x^{14}y^{14}$$

Randy, I use the word "genus" here. Happy?

For $n \geq 5$, the intersection $\cap_{r=1}^{n-2} M_{n,r}$ has positive genus and so has no polynomial parameterization: despite the Theorem, there are no versions of Desboves-Elkies in higher degrees.

I spent some time searching for other "interesting" identities of this kind and found this one:

$$\sum_{k=0}^{4} (\zeta_5^k x^2 + a\ x\ y + \zeta_5^{-k})^{14} =$$

$$f(a)(x^{24}y^4 + x^4y^{24}) + g(a)(x^{19}y^9 + x^9y^{19}) + h(a)x^{14}y^{14}$$

$$\text{where} \quad f(a) = 455(1+a^2)(1+11a^2) \quad \text{and}$$

$$g(a) = 10010a(1+a^2)(5+25a^2+11a^4+a^6).$$

Miraculously, $f(i) = g(i) = 0$, and $h(i) = 5^7$.

Again with the 14-th powers?

It follows that if $f_k(x,y) = \zeta_5^k x^2 + i\ x\ y + \zeta_5^{-k} y^2$ for $0 \le k \le 4$ and $f_5(x,y) = \sqrt{-5}\ x\ y$, then

$$\sum_{j=0}^{5} f_j^{14}(x,y) = 0. \tag{10}$$

Again with the 14-th powers?

It follows that if $f_k(x,y) = \zeta_5^k x^2 + i\ x\ y + \zeta_5^{-k} y^2$ for $0 \le k \le 4$ and $f_5(x,y) = \sqrt{-5}\ x\ y$, then

$$\sum_{j=0}^{5} f_j^{14}(x,y) = 0. \tag{10}$$

Mark Green has shown that if r *entire* (let alone polynomial) functions ϕ_j satisfy $\sum_{j=1}^{r} \phi_j^N = 0$, then $N \le r(r-2)$; 14 is not that much less than 24.

Again with the 14-th powers?

It follows that if $f_k(x,y) = \zeta_5^k x^2 + i\ x\ y + \zeta_5^{-k} y^2$ for $0 \le k \le 4$ and $f_5(x,y) = \sqrt{-5}\ x\ y$, then

$$\sum_{j=0}^{5} f_j^{14}(x,y) = 0. \tag{10}$$

Mark Green has shown that if r *entire* (let alone polynomial) functions ϕ_j satisfy $\sum_{j=1}^{r} \phi_j^N = 0$, then $N \le r(r-2)$; 14 is not that much less than 24.
I don't know **why** (10) is true. Possible hint:

$$\sum_{j=0}^{5} f_j^{2k}(x,y) = 0 \quad \text{for} \quad k = 1,2,4$$

and $M_{6,1} = M_{6,2} = M_{6,4} = 0 \implies M_{6,7} = 0$.
The question is: **why** do the f_j^2's lie on this intersection?

Liouville, not Louisville, please!

Here's (4) again:

$$\sum_{1\le i<j\le 4} (x_i + x_j)^4 + (x_i - x_j)^4 = 6(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2$$

This can be proved by noting that

$$(a+b)^4 + (a-b)^4 = 2a^4 + 12a^2b^2 + b^4$$

and counting the number of times a given monomial occurs on each side.

Liouville, not Louisville, please!

Here's (4) again:

$$\sum_{1\le i<j\le 4} (x_i + x_j)^4 + (x_i - x_j)^4 = 6(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2$$

This can be proved by noting that

$$(a+b)^4 + (a-b)^4 = 2a^4 + 12a^2b^2 + b^4$$

and counting the number of times a given monomial occurs on each side.

Liouville used a version of this in 1859 to make the first advance on Waring's Problem since Lagrange's Four-Square Theorem.

Theorem

Every positive integer n is a sum of at most 53 4-th powers of integers.

Look up “zenzizenzic” in the OED

Proof.

Write $n = t + 6m$, where $0 \le t \le 5$. By Lagrange, write $m = \sum_{i=1}^{4} x_i^2$, and then write $x_i = \sum_{j=1}^{4} y_{ij}^2$. Then

$$n = t + \sum_{i=1}^{4} 6(y_{i1}^2 + y_{i2}^2 + y_{i3}^2 + y_{i4}^2)^2$$

Look up “zenzizenzic” in the OED

Proof.

Write $n = t + 6m$, where $0 \le t \le 5$. By Lagrange, write $m = \sum_{i=1}^{4} x_i^2$, and then write $x_i = \sum_{j=1}^{4} y_{ij}^2$. Then

$$n = t + \sum_{i=1}^{4} 6(y_{i1}^2 + y_{i2}^2 + y_{i3}^2 + y_{i4}^2)^2$$

Look up "zenzizenzic" in the OED

Proof.

Write $n = t + 6m$, where $0 \le t \le 5$. By Lagrange, write $m = \sum_{i=1}^4 x_i^2$, and then write $x_i = \sum_{j=1}^4 y_{ij}^2$. Then

$$n = t + \sum_{i=1}^{4} 6(y_{i1}^2 + y_{i2}^2 + y_{i3}^2 + y_{i4}^2)^2$$

which by (4) is a sum of $t \le 5$ copies of 1^4 and 4×12 summands of the form $(y_{ij} \pm y_{ik})^4$. □

Look up "zenzizenzic" in the OED

Proof.

Write $n = t + 6m$, where $0 \le t \le 5$. By Lagrange, write $m = \sum_{i=1}^{4} x_i^2$, and then write $x_i = \sum_{j=1}^{4} y_{ij}^2$. Then

$$n = t + \sum_{i=1}^{4} 6(y_{i1}^2 + y_{i2}^2 + y_{i3}^2 + y_{i4}^2)^2$$

which by (4) is a sum of $t \le 5$ copies of 1^4 and 4×12 summands of the form $(y_{ij} \pm y_{ik})^4$. □

For example, $1859 = 5 + 6 * 309 = 5 + 6 * (16^2 + 6^2 + 4^2 + 1^2)$ is one such representation, and after writing $16, 6, 4, 1$ each as a sum of squares, one is led to

$$1859 = 6 \cdot 4^4 + 2 \cdot 3^4 + 9 \cdot 2^4 + 17 \cdot 1^4 + 19 \cdot 0^4.$$

I didn't mean look it up during the talk, later. Later!

This is not the best way to study Waring's problem, and 53 is far from optimal. (For example, $1859 = 6^4 + 2 * 4^4 + 3 * 2^4 + 3 * 1^4$, with 9 cubes.)

I didn't mean look it up during the talk, later. Later!

This is not the best way to study Waring's problem, and 53 is far from optimal. (For example, $1859 = 6^4 + 2 * 4^4 + 3 * 2^4 + 3 * 1^4$, with 9 cubes.)

Mathematicians in the rest of the 19th century gave similar formulas for degrees 6, 8 and 10 and then, as usual, Hilbert destroyed their cottage industry when he solved Waring's Problem in 1909. A key step was this non-constructive theorem:

I didn't mean look it up during the talk, later. Later!

This is not the best way to study Waring's problem, and 53 is far from optimal. (For example, $1859 = 6^4 + 2 * 4^4 + 3 * 2^4 + 3 * 1^4$, with 9 cubes.)

Mathematicians in the rest of the 19th century gave similar formulas for degrees 6, 8 and 10 and then, as usual, Hilbert destroyed their cottage industry when he solved Waring's Problem in 1909. A key step was this non-constructive theorem:

Theorem

For all n, r, *let* $N = \binom{n+2r-1}{n-1}$. *Then there exist* $0 < \lambda_k \in \mathbb{Q}$ *and* $\alpha_{kj} \in \mathbb{Z}, 1 \le k \le N, 1 \le j \le n$, *such that*

$$\sum_{k=1}^{N} \lambda_k(\alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n)^{2r} = (x_1^2 + \cdots + x_n^2)^r$$

The basic idea is of the proof to find the “average” $2r$-th power, where the coefficients range over the unit sphere S^{n-1}, by computing

$$F_{2r}(S^{n-1},\mu;x) := \int_{u\in S^{n-1}} (u_1x_1 + \cdots + u_nx_n)^{2r}\,d\mu$$

where μ is the unit rotation-invariant measure.

The basic idea is of the proof to find the "average" $2r$-th power, where the coefficients range over the unit sphere S^{n-1}, by computing

$$F_{2r}(S^{n-1},\mu;x) := \int_{u\in S^{n-1}} (u_1x_1+\cdots+u_nx_n)^{2r}\,d\mu$$

where μ is the unit rotation-invariant measure.
If $a, b \in \mathbb{R}^n$ and $||a|| = ||b||$, then by the rotational invariance, $F_{2r}(S^{n-1},\mu;a) = F_{2r}(S^{n-1},\mu;b)$.

The basic idea is of the proof to find the "average" $2r$-th power, where the coefficients range over the unit sphere S^{n-1}, by computing

$$F_{2r}(S^{n-1},\mu;x) := \int_{u\in S^{n-1}} (u_1x_1+\cdots+u_nx_n)^{2r}\,d\mu$$

where μ is the unit rotation-invariant measure.
If $a, b \in \mathbb{R}^n$ and $||a|| = ||b||$, then by the rotational invariance, $F_{2r}(S^{n-1},\mu;a) = F_{2r}(S^{n-1},\mu;b)$.
Thus $F_{2r}(S^{n-1},\mu;x)$ is a function of $||x||$ and since it is a form in the x_j's of degree $2r$,

$$F_{2r}(S^{n-1},\mu;x) = c_{n,r}(x_1^2+\cdots+x_n^2)^r$$

for some positive constant $c_{n,r}$. This constant can be computed by choosing x to be a unit vector and doing the integral.

When's the average talk over, already?

The next step is approximate the integral with a Riemann sum and use Carathéodory's Theorem to show that each such sum can be replaced by one with at most N terms. Ultimately, an application Bolzano-Weierstrass gives a convergent subsequence. The argument that the coefficients are rational is subtle!

When's the average talk over, already?

The next step is approximate the integral with a Riemann sum and use Carathéodory's Theorem to show that each such sum can be replaced by one with at most N terms. Ultimately, an application Bolzano-Weierstrass gives a convergent subsequence. The argument that the coefficients are rational is subtle!

I want to finish by giving some applications of these *Hilbert identities*. It is sometimes convenient to ignore the algebraic constraints, and absorb the λ_k's into the powers by writing

$$(\beta_{k1}x_1 + \cdots + \beta_{kn}x_n)^{2r} = \lambda_k(\alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n)^{2r}.$$

I'll start skipping some of the old slides

The rest of the talk will give some applications. Suppose

$$\sum_{k=1}^{N}(\beta_{k1}x_1 + \cdots + \beta_{kn}x_n)^{2r} = (x_1^2 + \cdots + x_n^2)^r.$$

Dvoretzky's Theorem in functional analysis says that any infinite-dimensional Banach space contains isometric copies of every ℓ_2. Hilbert Identities can be used for concrete finite-dimensional examples.

For example, consider the vectors $u_j = (\beta_{1j}, \dots, \beta_{Nj}) \in \mathbb{R}^N$, $1 \le j \le n$. For any $x \in \mathbb{R}^n$, $||\sum_j x_j u_j||_{2r}^{2r}$ is the left side of (16), which by the right side is $||x||_2^{2r}$. In other words, $< u_j > \subset \ell_{2r}^N$ is an n dimensional subspace which is isometric to ℓ_2^n.

Suppose a set S and non-negative measure μ are given. An *exact quadrature formula for* (S,μ) *of degree* d is an expression

$$\int_{u\in S} p(u)\ d\mu = \sum_{k=1}^{N} \lambda_k p(\alpha_k),$$

which holds for **all** forms p of degree d. (Traditionally, $\alpha_k \in S$ and $\lambda_k \geq 0$.)

Suppose a set S and non-negative measure μ are given. An *exact quadrature formula for* (S,μ) *of degree* d is an expression

$$\int_{u\in S} p(u)\ d\mu = \sum_{k=1}^{N} \lambda_k p(\alpha_k),$$

which holds for **all** forms p of degree d. (Traditionally, $\alpha_k \in S$ and $\lambda_k \geq 0$.) Such an equation holds if and only if it holds for all monomials: $x^i = x_1^{i_1} \cdots x_n^{i_n}$ of degree d. In this case, we have

$$\int_{u\in S} u^i\ d\mu = \sum_{k=1}^{N} \lambda_k \alpha_{k1}^{i_1} \dots \alpha_{kn}^{i_n}$$

Suppose a set S and non-negative measure μ are given. An *exact quadrature formula for* (S,μ) *of degree* d is an expression

$$\int_{u\in S} p(u)\ d\mu = \sum_{k=1}^{N} \lambda_k p(\alpha_k),$$

which holds for **all** forms p of degree d. (Traditionally, $\alpha_k \in S$ and $\lambda_k \geq 0$.) Such an equation holds if and only if it holds for all monomials: $x^i = x_1^{i_1}\cdots x_n^{i_n}$ of degree d. In this case, we have

$$\int_{u\in S} u^i\ d\mu = \sum_{k=1}^{N} \lambda_k \alpha_{k1}^{i_1}\ldots\alpha_{kn}^{i_n}$$

If we now multiply by $\frac{d!}{i_1!\ldots i_n!}x^i$ and sum, we get

$$\int_{u\in S} (x_1u_1+\cdots+x_nu_n)^d\ d\mu = \sum_{k=1}^{N} \lambda_k(\alpha_{k1}x_1+\cdots+\alpha_{kn}x_n)^d.$$

If you'll trust me with the constants, we now have the following:

Theorem

Suppose μ *is the rotation-invariant unit measure on* S^{n-1} *and* $\lambda_k \in \mathbb{R}$, $\alpha_k \in \mathbb{R}^n$. *Then*

$$\int_{u \in S^{n-1}} p(u)\ d\mu = \sum_{k=1}^{N} \lambda_k p(\alpha_k),$$

is an exact quadrature formula of degree d *for* (S^{n-1}, μ) *iff*

$$\sum_{k=1}^{N} \lambda_k (\alpha_{k1} x_1 + \cdots + \alpha_{kn} x_n)^d = c_{n,d} (x_1^2 + \cdots + x_n^2)^{d/2},$$

where $c_{n,2r} = \prod_{j=1}^{r} \frac{n+2j}{1+2j}$ *and* $c_{n,2r+1} = 0$.

If q is a form of degree $d-2i$, then $(\sum x_j^2)^i q$ is a form of degree d which agrees with q on S^{n-1}, so an exact quadrature formula of degree d is also one of degree $d-2i$. If d is odd, the integral vanishes. By writing f as a sum of homogeneous pieces, we get

If q is a form of degree $d-2i$, then $(\sum x_j^2)^i q$ is a form of degree d which agrees with q on S^{n-1}, so an exact quadrature formula of degree d is also one of degree $d-2i$. If d is odd, the integral vanishes. By writing f as a sum of homogeneous pieces, we get

Corollary

If

$$\int_{u\in S^{n-1}} p(u)\ d\mu = \sum_{k=1}^{N} \lambda_k p(\alpha_k),$$

is an exact quadrature formula of degree d, then for **every** *polynomial f (homogeneous or not) of degree $\le 2\lfloor \frac{d}{2} \rfloor + 1$,*

$$\int_{u\in S^{n-1}} f(u)\ d\mu = \sum_{k=1}^{N} \frac{\lambda_k}{2}\big(f(\alpha_k)+f(-\alpha_k)\big)$$

These establish the centrality of Hilbert Identities for quadrature formulas on S^{n-1}. Another corollary uses an old ~~trick~~ method from numerical analysis.

Corollary

In any Hilbert Identity, $N \geq \binom{n+r-1}{n-1}$.

These establish the centrality of Hilbert Identities for quadrature formulas on S^{n-1}. Another corollary uses an old ~~trick~~ method from numerical analysis.

Corollary

In any Hilbert Identity, $N \geq \binom{n+r-1}{n-1}$.

Proof.

If $N < \binom{n+r-1}{r-1}$, then there exists a non-zero form h of degree r so that $h(\alpha_k) = 0$, $1 \leq k \leq N$. (Count the number of monomials.) Now put $p = h^2$ into the quadrature formula; we have

$$\int_{u \in S^{n-1}} h^2(u)\ d\mu = \sum_{k=1}^{N} \lambda_k h^2(\alpha_k),$$

which is > 0 on the left, and 0 on the right. Contradiction! ☐

Sorry, I dozed off. Is he still talking?

If a Hilbert Identity has minimal length, then the summands have some special properties

Corollary

If

$$\sum_{k=1}^{N}(\beta_{k1}x_1 + \cdots + \beta_{kn}x_n)^{2r} = (x_1^2 + \cdots + x_n^2)^r.$$

and $N = \binom{n+r-1}{n-1}$ *is minimal, then*

$$\left(\sum_{\ell=1}^{n}\beta_{k\ell}^2\right)^r = \frac{1}{N}\prod_{j=1}^{r}\frac{n+2j}{1+2j}$$

is independent of k.

I prefer my cones to be closed, convex and chocolate

Here's a sketch of the proof. Let $Q_{n,2r}$ denote the (closed) convex cone of sums of $2r$-th powers of linear forms. Hilbert's Theorem implies that $(\sum x_j^2)^r$ is interior to the cone, and there exists $c > 0$ so that $(\sum x_j^2)^r - cx_1^{2r}$ is on the boundary of the cone $Q_{n,2r}$.

I prefer my cones to be closed, convex and chocolate

Here's a sketch of the proof. Let $Q_{n,2r}$ denote the (closed) convex cone of sums of $2r$-th powers of linear forms. Hilbert's Theorem implies that $(\sum x_j^2)^r$ is interior to the cone, and there exists $c > 0$ so that $(\sum x_j^2)^r - cx_1^{2r}$ is on the boundary of the cone $Q_{n,2r}$. Each $(x_1^2 + \cdots + x_n^2)^r - (\beta_{k1}x_1 + \cdots + \beta_{kn}x_n)^{2r}$ is on the boundary of the cone, because N is minimal. Now rotate the variables so that this polynomial becomes $(x_1^2 + \cdots + x_n^2)^r - (\sum_\ell \beta_{k\ell}^2)^r x_1^{2r}$.

I prefer my cones to be closed, convex and chocolate

Here's a sketch of the proof. Let $Q_{n,2r}$ denote the (closed) convex cone of sums of $2r$-th powers of linear forms. Hilbert's Theorem implies that $(\sum x_j^2)^r$ is interior to the cone, and there exists $c > 0$ so that $(\sum x_j^2)^r - cx_1^{2r}$ is on the boundary of the cone $Q_{n,2r}$. Each $(x_1^2 + \cdots + x_n^2)^r - (\beta_{k1}x_1 + \cdots + \beta_{kn}x_n)^{2r}$ is on the boundary of the cone, because N is minimal. Now rotate the variables so that this polynomial becomes $(x_1^2 + \cdots + x_n^2)^r - (\sum_\ell \beta_{k\ell}^2)^r x_1^{2r}$.

When $n = 2$ and $d > r$, we've already seen that the vertices of a regular $2d$-gon (with constant weights) give an exact quadrature formula for polynomials of degree $\le 2r + 1$, using $r + 1 = \binom{2+r-1}{2-1}$ summands.

“Final” is one of my favorite words!

This leads to the final interpretation of Hilbert Identities. In a beautiful series of papers in the 1970s, Delsarte, Goethals and Seidel introduced the idea of the spherical design.

“Final” is one of my favorite words!

This leads to the final interpretation of Hilbert Identities. In a beautiful series of papers in the 1970s, Delsarte, Goethals and Seidel introduced the idea of the spherical design.
A set $X = \{v_1, \dots, v_N\} \in \mathbb{R}^n$ is a *spherical t-design* if for every polynomial $p(x_1, \dots, x_n)$, deg $p \le t$, we have

$$\frac{\int_{S^{n-1}} f(x)\ d\mu}{\int_{S^{n-1}} d\mu} = \frac{1}{N}\sum_{j=1}^{N} f(v_j).$$

"Final" is one of my favorite words!

This leads to the final interpretation of Hilbert Identities. In a beautiful series of papers in the 1970s, Delsarte, Goethals and Seidel introduced the idea of the spherical design.
A set $X = \{v_1, \dots, v_N\} \in \mathbb{R}^n$ is a *spherical t-design* if for every polynomial $p(x_1, \dots, x_n)$, $\deg p \le t$, we have

$$\frac{\int_{S^{n-1}} f(x)\ d\mu}{\int_{S^{n-1}} d\mu} = \frac{1}{N} \sum_{j=1}^{N} f(v_j).$$

That is, the average of p on the sphere is equal to the average of p on these points. There are some wonderful theorems about spherical designs.

I thought he said "final"

- The vertices of a regular d-gon are a spherical t-design in $\mathbb{R}^2$ if $d > t$.

I thought he said "final"

- The vertices of a regular d-gon are a spherical t-design in $\mathbb{R}^2$ if $d > t$.
- For all (n, t), there exist spherical t-designs in $\mathbb{R}^n$.

I thought he said "final"

- The vertices of a regular d-gon are a spherical t-design in $\mathbb{R}^2$ if $d > t$.
- For all (n, t), there exist spherical t-designs in $\mathbb{R}^n$.
- If $t = 2s$, then $N \geq \binom{n+s-1}{n-1} + \binom{n+s-2}{n-1}$; if $t = 2s + 1$, then $N \geq 2\binom{n+s-1}{n-1}$, and there exists $N(n, t)$ so that for all $N \geq N(n, t)$, such a t-design with N points exists (Seymour and Zaslavsky).

I thought he said “final”

- The vertices of a regular d-gon are a spherical t-design in $\mathbb{R}^2$ if $d > t$.
- For all (n, t), there exist spherical t-designs in $\mathbb{R}^n$.
- If $t = 2s$, then $N \geq \binom{n+s-1}{n-1} + \binom{n+s-2}{n-1}$; if $t = 2s + 1$, then $N \geq 2\binom{n+s-1}{n-1}$, and there exists $N(n, t)$ so that for all $N \geq N(n, t)$, such a t-design with N points exists (Seymour and Zaslavsky).
- If $d = 2s + 1$ and $N = 2\binom{n+s-1}{n-1}$, then X is called a *tight* spherical design. Such a tight spherical design must be antipodal and so its coefficients give a Hilbert Identity of minimal length.

I thought he said "final"

- The vertices of a regular d-gon are a spherical t-design in $\mathbb{R}^2$ if $d > t$.
- For all (n, t), there exist spherical t-designs in $\mathbb{R}^n$.
- If $t = 2s$, then $N \geq \binom{n+s-1}{n-1} + \binom{n+s-2}{n-1}$; if $t = 2s + 1$, then $N \geq 2\binom{n+s-1}{n-1}$, and there exists $N(n, t)$ so that for all $N \geq N(n, t)$, such a t-design with N points exists (Seymour and Zaslavsky).
- If $d = 2s + 1$ and $N = 2\binom{n+s-1}{n-1}$, then X is called a *tight* spherical design. Such a tight spherical design must be antipodal and so its coefficients give a Hilbert Identity of minimal length.
- Your favorite symmetric pointset in $\mathbb{R}^n$ is a spherical design.

I thought he said “final”

- The vertices of a regular d-gon are a spherical t-design in $\mathbb{R}^2$ if $d > t$.
- For all (n, t), there exist spherical t-designs in $\mathbb{R}^n$.
- If $t = 2s$, then $N \geq \binom{n+s-1}{n-1} + \binom{n+s-2}{n-1}$; if $t = 2s+1$, then $N \geq 2\binom{n+s-1}{n-1}$, and there exists $N(n,t)$ so that for all $N \geq N(n,t)$, such a t-design with N points exists (Seymour and Zaslavsky).
- If $d = 2s+1$ and $N = 2\binom{n+s-1}{n-1}$, then X is called a *tight* spherical design. Such a tight spherical design must be antipodal and so its coefficients give a Hilbert Identity of minimal length.
- Your favorite symmetric pointset in $\mathbb{R}^n$ is a spherical design.
- A tight spherical $2s+1$-design in $\mathbb{R}^n$ defines the maximal number of lines through the origin in $\mathbb{R}^n$ which make only s different angles with each other.

- Tight spherical $2s+1$-designs exist whenever $n = 2$ and $2s+1 = 3$ and for $(2s+1, n) =$ (5,7), (5,23), (7,8), (7,23), (11,24). Otherwise, they are impossible unless $2s+1 = 5$ and $n = u^2 - 2$ (u odd) or $2s+1 = 7$ and $n = 3v^2 - 4$. Some non-existence results exist, but many cases remain open.

- Tight spherical $2s+1$-designs exist whenever $n=2$ and $2s+1=3$ and for $(2s+1,n) =$ (5,7), (5,23), (7,8), (7,23), (11,24). Otherwise, they are impossible unless $2s+1=5$ and $n=u^2-2$ (u odd) or $2s+1=7$ and $n=3v^2-4$. Some non-existence results exist, but many cases remain open.
- No new tight spherical designs have been found in the last 30 years. All known tight spherical designs are unique, up to rotation. All known proofs of this are *ad hoc*.

- Tight spherical $2s+1$-designs exist whenever $n = 2$ and $2s + 1 = 3$ and for $(2s + 1, n) =$ (5,7), (5,23), (7,8), (7,23), (11,24). Otherwise, they are impossible unless $2s + 1 = 5$ and $n = u^2 - 2$ (u odd) or $2s + 1 = 7$ and $n = 3v^2 - 4$. Some non-existence results exist, but many cases remain open.
- No new tight spherical designs have been found in the last 30 years. All known tight spherical designs are unique, up to rotation. All known proofs of this are *ad hoc*.
- Tight spherical designs lead to beautiful Hilbert Identities, as in (4). Take the indices below as cyclic mod 7, then

$$\sum_{i=1}^{7} \sum_{\pm} (x_i \pm x_{i+1} \pm x_{i+3})^4 = 12(x_1^2 + \cdots + x_7^2)^2.$$

This comes from the finite projective plane of order 2.

It's not a number theory talk without the Golden Ratio

- The tight 11-design in $\mathbb{R}^{24}$ is derived from the minimal vectors in the Leech lattice and has the following hilarious implication. There is an isometric copy of ℓ_2^{24} in ℓ_{10}^{98280}, but not in ℓ_{10}^{98279}.

It's not a number theory talk without the Golden Ratio

- The tight 11-design in $\mathbb{R}^{24}$ is derived from the minimal vectors in the Leech lattice and has the following hilarious implication. There is an isometric copy of ℓ_2^{24} in ℓ_{10}^{98280}, but not in ℓ_{10}^{98279}.
- Using the Schönemann coordinates for an icosahedron and letting $\Phi = \frac{\sqrt{5}+1}{2}$, so that $\Phi^4 + 1 = 3\Phi^2$, we have

$$6\Phi^2(x^2+y^2+z^2)^2 = (\Phi x + y)^4 + (\Phi x - y)^4 +$$
$$(\Phi y + z)^4 + (\Phi y - z)^4 + (\Phi z + x)^4 + (\Phi z - x)^4.$$

The speaker's favorite theorem

Here's an identity which combines the previous discussion with most of your favorite small integers.

Theorem

If the equation

$$(x_1^2 + x_2^2 + x_3^2)^2 = \sum_{k=1}^{r} (a_k x_1 + b_k x_2 + c_k x_3)^4 \tag{11}$$

holds, then $r \geq 6$. *If* $r = 6$, *then this equation is true if and only if the 12 points* $\pm(a_k, b_k, c_k)$ *are the vertices of a regular icosahedron inscribed in a sphere with center 0 and radius* $(5/6)^{1/4}$.

The speaker's favorite theorem

Here's an identity which combines the previous discussion with most of your favorite small integers.

Theorem

If the equation

$$(x_1^2 + x_2^2 + x_3^2)^2 = \sum_{k=1}^{r} (a_k x_1 + b_k x_2 + c_k x_3)^4 \tag{12}$$

holds, then $r \geq 6$. *If* $r = 6$, *then this equation is true if and only if the 12 points* $\pm(a_k, b_k, c_k)$ *are the vertices of a regular icosahedron inscribed in a sphere with center 0 and radius* $(5/6)^{1/4}$.

References

For (1) and (4):

- Dickson, L. E., History of the Theory of Numbers, v.2., AMS (1920, 1966)

References

For (1) and (4):

- Dickson, L. E., History of the Theory of Numbers, v.2., AMS (1920, 1966)
- Friedman, A., Mean values and polyharmonic polynomials, Michigan Math. J. 4 (1957), 137–145

References

For (1) and (4):

- Dickson, L. E., History of the Theory of Numbers, v.2., AMS (1920, 1966)
- Friedman, A., Mean values and polyharmonic polynomials, Michigan Math. J. 4 (1957), 137–145
- Ellison, W. J., Waring's problem, Amer. Math. Monthly 78 (1971), no. 1, 10–36

References

For (1) and (4):

- Dickson, L. E., History of the Theory of Numbers, v.2., AMS (1920, 1966)
- Friedman, A., Mean values and polyharmonic polynomials, Michigan Math. J. 4 (1957), 137–145
- Ellison, W. J., Waring's problem, Amer. Math. Monthly 78 (1971), no. 1, 10–36
- Delsarte, P, Goethals, J.-M., Seidel, J. J., Spherical codes and designs, Geometriae Dedicata 6 (1977), no. 3, 363–388, and many others

References

For (1) and (4):

- Dickson, L. E., History of the Theory of Numbers, v.2., AMS (1920, 1966)
- Friedman, A., Mean values and polyharmonic polynomials, Michigan Math. J. 4 (1957), 137–145
- Ellison, W. J., Waring's problem, Amer. Math. Monthly 78 (1971), no. 1, 10–36
- Delsarte, P, Goethals, J.-M., Seidel, J. J., Spherical codes and designs, Geometriae Dedicata 6 (1977), no. 3, 363–388, and many others
- Sums of even powers of real linear forms, Mem. AMS, No. 463, 1992

References

For (2):

References

For (2):

- Viète, F., The Analytic Art, translated by T. Richard Witmer (1591, 1983)

References

For (2):

- Viète, F., The Analytic Art, translated by T. Richard Witmer (1591, 1983)
- (with J. Rouse) On the sums of two cubes, to appear in the Int. J. Number Theory, available on my website or on the arXiv: 1012.5801

References

For (2):

- Viète, F., The Analytic Art, translated by T. Richard Witmer (1591, 1983)
- (with J. Rouse) On the sums of two cubes, to appear in the Int. J. Number Theory, available on my website or on the arXiv: 1012.5801

References

For (2):

- Viète, F., The Analytic Art, translated by T. Richard Witmer (1591, 1983)
- (with J. Rouse) On the sums of two cubes, to appear in the Int. J. Number Theory, available on my website or on the arXiv: 1012.5801

For (3):

References

For (2):

- Viète, F., The Analytic Art, translated by T. Richard Witmer (1591, 1983)
- (with J. Rouse) On the sums of two cubes, to appear in the Int. J. Number Theory, available on my website or on the arXiv: 1012.5801

For (3):

- Patterns of dependence among powers of polynomials, DIMACS Ser. in Discrete Mathematics and Theoretical Computer Science, 60 (2003), 101-121

Special thanks to my silicon friend, Mac, for his commentary.

Thank you all for listening. You have no idea what it's like to be shut up in a laptop all the time.

Thank you all for listening. You have no idea what it's like to be shut up in a laptop all the time.

Sure I get to travel, but I never see anything, he can't read the screen outdoors he says, yeah, that's what he says.

Thank you all for listening. You have no idea what it's like to be shut up in a laptop all the time.

Sure I get to travel, but I never see anything, he can't read the screen outdoors he says, yeah, that's what he says.

Hey, wait a minute, I'm not cut off here. You mean I could have said more than one line each page.

Thank you all for listening. You have no idea what it's like to be shut up in a laptop all the time.

Sure I get to travel, but I never see anything, he can't read the screen outdoors he says, yeah, that's what he says.

Hey, wait a minute, I'm not cut off here. You mean I could have said more than one line each page.

All that terseness for nothing! What a bummer!

Thank you all for listening. You have no idea what it's like to be shut up in a laptop all the time.

Sure I get to travel, but I never see anything, he can't read the screen outdoors he says, yeah, that's what he says.

Hey, wait a minute, I'm not cut off here. You mean I could have said more than one line each page.

All that terseness for nothing! What a bummer!

I'm going to remember this for the next time Brightshirt over here shleps me to a conference. You think the margin was too narrow for Fermat's "marvelous proof", wait until you see how wide I make the heading next time!