

# The secret lives of polynomial identities

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At first, they seem out of place, but enough reflection reveals that the best identities can signify deep and distant phenomena.



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Not bad for one identity.

Not all identities are interesting, of course. Sometimes they're just a consequence of linear dependence. For example, who cares that

$$(x + 2y)^2 + (2x + 3y)^2 + (3x + 4y)^2 = 14x^2 + 40xy + 29y^2 ?$$

The left hand side has to equal ... *some* binary quadratic form. Of course, identities based on dependence can become interesting if their coefficients have additional properties.

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Two examples are the binomial theorem and the formula for the  $n$ -th difference:

$$\sum_{k=0}^n \binom{n}{k} x^{n-k} y^k = (x + y)^n$$
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Identities are also interesting if there are many fewer summands than you'd expect.

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This talk consists of stories about some semi-astonishing identities: their deeper meanings and how they can (or maybe should) be derived.

The first identity must have its roots in 19th century mathematics, although in this explicit form, I've only been able to trace it back to the mid 1950s. It's one of a family, and it's not accidental in this version that  $(1^2 + (\sqrt{3})^2)^5 = 4^5 = 2^{10} = 1024$ :

$$1024x^{10} + 1024y^{10} + (x + \sqrt{3} y)^{10} + (x - \sqrt{3} y)^{10} + (\sqrt{3} x + y)^{10} + (\sqrt{3} x - y)^{10} = 1512(x^2 + y^2)^5 \quad (1)$$

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The story of (1) and (4) (a few slides from now) and their generalizations runs through (at least) number theory, numerical analysis, functional analysis and combinatorics.

The second identity is very old; it goes back to Viète:

$$x^3 + y^3 = \left( \frac{x^4 + 2xy^3}{x^3 - y^3} \right)^3 + \left( \frac{y^4 + 2x^3y}{y^3 - x^3} \right)^3, \quad (2)$$

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This is used to show that a sum of two cubes of rational numbers can usually be so expressed in infinitely many ways. For example:

$$2^3 + 1^3 = \left( \frac{20}{7} \right)^3 + \left( -\frac{17}{7} \right)^3 = \left( -\frac{36520}{90391} \right)^3 + \left( \frac{188479}{90391} \right)^3 = \dots$$



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The story here is a description of all homogeneous solutions to

$$x^3 + y^3 = p^3(x, y) + q^3(x, y), \quad p, q \in \mathbb{C}(x, y).$$

Viète's derivation of his identity, curiously, is formally identical to a common technique in the modern study of elliptic curves.

The third identity was independently found by Desboves (1880) and Elkies (1995):

$$(x^2 + \sqrt{2} x y - y^2)^5 + (i x^2 - \sqrt{2} x y + i y^2)^5 + (-x^2 + \sqrt{2} x y + y^2)^5 + (-i x^2 - \sqrt{2} x y - i y^2)^5 = 0. \quad (3)$$

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It was first discovered by observing that

$$\sum_{k=0}^3 (i^k x^2 + i^{2k} a x y + i^{3k} y^2)^5 = 40a(a^2 + 2)(x^7 y^3 + x^3 y^7),$$

and then setting  $a = \sqrt{-2}$  and  $y \rightarrow i y$ . But why  $\sqrt{-2}$ ? The full story ultimately depends on Newton's Theorem on symmetric polynomials. Commutative algebra and algebraic geometry also play a role, but Felix Klein would say it's all based on the cube.

The fourth identity was used by Liouville to show that every positive integer is a sum of at most 53 4th powers of integers:

$$\sum_{1 \leq i < j \leq 4} ((x_i + x_j)^4 + (x_i - x_j)^4) = 6(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2. \quad (4)$$

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Many similar and more complicated formulas were found in the late 19th century, until Hilbert showed that they must exist in all degrees.

As one indication of their geometric and combinatorial significance, if you take the coordinates of the coefficients of the  $2\binom{4}{2}$  linear forms in (4), together with their antipodes, you get the 24 points  $(\pm 1, \pm 1, 0, 0)$  and their permutations.

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These are the vertices of a regular polytope in  $\mathbb{R}^4$  called the *24-cell*.

One idea used a lot in this talk is, I hope, fairly familiar. Suppose  $2 \leq d \in \mathbb{N}$ . Let

$$\zeta_d = e^{\frac{2\pi i}{d}} = \cos\left(\frac{2\pi}{d}\right) + i \sin\left(\frac{2\pi}{d}\right)$$

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Since the sum below is a finite geometric progression, it is easy to see that

### Lemma

$$\sum_{r=0}^{d-1} (\zeta_d^k)^r = \begin{cases} d, & \text{if } d \mid k; \\ 0, & \text{otherwise.} \end{cases}$$

We'll use this lemma in sums of polynomials “synched” with powers of  $\zeta_d$ , so that only every  $d$ -th monomial can possibly occur.



Let's look at the first identity again and pull out a factor of  $2^{10}$ :

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It's looking better already. You may recognize this as

$$\sum_{j=0}^5 \left( \cos \left( \frac{j\pi}{6} \right) x + \sin \left( \frac{j\pi}{6} \right) y \right)^{10} = \frac{189}{128} (x^2 + y^2)^5.$$

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(The first explicit appearance I've found of the underlying general theorem below is in a paper of Avner Friedman (1957).)

### Theorem

*If  $d > r$ , then for all  $\theta$ ,*

$$\begin{aligned} \sum_{j=0}^{d-1} \left( \cos \left( \frac{j\pi}{d} + \theta \right) x + \sin \left( \frac{j\pi}{d} + \theta \right) y \right)^{2r} \\ = \frac{d}{2^{2r}} \binom{2r}{r} (x^2 + y^2)^r \end{aligned} \tag{5}$$

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Put  $d = 6$ ,  $r = 5$  and  $\theta = 0$  in (5); then  $\frac{6}{2^{10}} \binom{10}{5} = \frac{6 \cdot 252}{1024}$   
 $= \frac{1512}{1024} = \frac{189}{128}$ .

The fastest proof of the Theorem is to derive it from another formula, which uses synching at its best. Expand the left-hand side below, switch the order of summation and recall that  $\zeta_{2d}^{2m} = \zeta_d^m$ .

$$\sum_{j=0}^{d-1} (\zeta_{2d}^j u + \zeta_{2d}^{-j} v)^{2r}$$



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As we've seen, the inner sum is zero unless  $d \mid r - k$ . Since  $d > r$ , the only multiple of  $d$  in  $\{-r, -(r-1), \dots, 0, \dots, r-1, r\}$  is 0, corresponding to  $k = r$ , so

$$\sum_{j=0}^{d-1} (\zeta_{2d}^j u + \zeta_{2d}^{-j} v)^{2r} = d \binom{2r}{r} u^r v^r.$$

Finally, we substitute  $u = \frac{1}{2}e^{i\theta}(x + \frac{y}{i})$  and  $v = \frac{1}{2}e^{-i\theta}(x - \frac{y}{i})$  into

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$$u v = \frac{x^2 + y^2}{4},$$

and this proves the Theorem.

It's worth looking at the formula again and making it asymmetric:

$$d \geq r + 1 \implies \sum_{j=0}^{d-1} \zeta_{2d}^{2jr} (u + \zeta_{2d}^{-2j} v)^{2r} = d \binom{2r}{r} u^r v^r.$$

$$\sum_{j=0}^{d-1} \zeta_d^{jr} (u + \zeta_d^{-j} v)^{2r} = d \binom{2r}{r} u^r v^r.$$

The simplest possible case is  $r = 1$  and  $d = 2$  and  $\zeta_2 = -1$ . This becomes

$$(u + v)^2 - (u - v)^2 = 2 \binom{2}{1} uv = 4uv$$

If you set  $u = m^2$ ,  $v = n^2$  and transpose, you get the shape of the familiar parameterization of Pythagorean triples:

$$(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2.$$



Every 19th century math major knew that  $\tan\left(\frac{\pi}{8}\right) = \sqrt{2} - 1$ , so if we take  $r = 7$  and  $d = 8$  in the Theorem and let  $\lambda = 338 + 239\sqrt{2}$  and  $\alpha = \sqrt{2} - 1$ , and do some minor bookkeeping, we get

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It's actually useful, in a version that goes back to the 1860's.

## Corollary

If  $d > r$ ,  $\theta \in \mathbb{R}$  is arbitrary and  $p(x, y)$  is a polynomial with degree  $\leq 2r + 1$ , then

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} p(\cos t, \sin t) dt \\ &= \frac{1}{2d} \sum_{j=0}^{2d-1} p\left(\cos\left(\frac{j\pi}{d} + \theta\right), \sin\left(\frac{j\pi}{d} + \theta\right)\right). \end{aligned}$$

There are similar formulas in  $n > 2$  variables, as we'll see later. The main reason these are less explicit than for two variables is this: 2013 points placed evenly on a circle clearly should be the vertices of a regular 2013-gon. How should you place 2013 points “evenly” on the surface of  $S^{n-1}$ ?

In 1591 (or 1593), François Viète published a revolutionary work on algebra which has been translated into English as *The Analytic Art* by T. R. Witmer. Viète's "Zetetic XVIII" is

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I'll quote Viète's proof on the next page. Keep in mind that he was working at the dawn of algebra, when mathematicians were not yet comfortable with negative numbers and the algebraic conventions were very fluid. Viète used vowels as variables and consonants as constants.

“Let the two given cubes be  $B^3$  and  $D^3$ , the first to be greater and the second to be smaller. Two other cubes are to be found, the sum of which is equal to  $B^3 - D^3$ . Let  $B - A$  be the root of the first one that is to be found, and let  $B^2A/D^2 - D$  be the root of the second. Forming the cubes and comparing them with  $B^3 - D^3$ , it will be found that  $3D^3B/(B^3 + D^3)$  equals  $A$ . The root of the first cube to be found, therefore, is  $[B(B^3 - 2D^3)]/(B^3 + D^3)$  and of the second is  $[D(2B^3 - D^3)]/(B^3 + D^3)$ . And the sum of the two cubes of these is equal to  $B^3 - D^3$ .”



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That is,

$$B^3 - D^3 = \left( \frac{B(B^3 - 2D^3)}{B^3 + D^3} \right)^3 + \left( \frac{D(2B^3 - D^3)}{B^3 + D^3} \right)^3.$$

By setting  $B = x$  and  $D = -y$ , Viète's formula becomes (2):

$$x^3 + y^3 = \left( \frac{x(x^3 + 2y^3)}{x^3 - y^3} \right)^3 + \left( \frac{y(y^3 + 2x^3)}{y^3 - x^3} \right)^3 .$$

We'll explain later just how smart Viète was to choose the coefficients he used.

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Jeremy Rouse and I have recently written a paper in the IJNT in which we examine the more general equation

$$x^3 + y^3 = p^3(x, y) + q^3(x, y) \tag{6}$$

for homogeneous rational functions  $p, q \in \mathbb{C}(x, y)$ . You can find it on the arXiv or on my home page.

To examine this equation, we take a common denominator for the rational functions  $p, q$  and rewrite as:

$$x^3 + y^3 = \left( \frac{f(x, y)}{h(x, y)} \right)^3 + \left( \frac{g(x, y)}{h(x, y)} \right)^3$$

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$$\begin{aligned}x^3 + y^3 &= \left(\frac{f(x, y)}{h(x, y)}\right)^3 + \left(\frac{g(x, y)}{h(x, y)}\right)^3 \\ \implies h(x, y)^3(x^3 + y^3) &= f(x, y)^3 + g(x, y)^3.\end{aligned}\tag{7}$$

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It follows that if  $\pi(x, y)$  is irreducible and  $\pi$  divides any two of  $\{f, g, h\}$ , then it divides the third. Also note that  $f$  and  $g$  may be permuted and cube roots of unity  $\omega^j$  may appear. Assume that  $f, g, h$  are forms (that is, homogeneous). If  $\deg f = \deg g = d$ , then we call (7) a *solution of degree  $d$* .

Here is a roster of all the solutions of degree  $\leq 11$ .

There's an obvious solution of degree 1:  $(f, g, h) = (x, y, 1)$ .

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$$\begin{aligned} & (\zeta u + \zeta^{-1} v)^3 + (\zeta^{-1} u + \zeta v)^3 \\ &= (\zeta^3 + \zeta^{-3})(u^3 + v^3) + 3(\zeta + \zeta^{-1})(u^2 v + uv^2) \end{aligned}$$

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After  $(u, v) \mapsto (x^3, y^3)$ , this rearranges to:

$$x^3 + y^3 = \left( \frac{\zeta x^3 + \zeta^{-1} y^3}{\sqrt{3}xy} \right)^3 + \left( \frac{\zeta^{-1} x^3 + \zeta y^3}{\sqrt{3}xy} \right)^3. \quad (8)$$

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Let's call this the *small* solution.

There are two solutions of degree 7 which are complex conjugates of each other. Here's one of them.

$$f(x, y) = x(x^6 + (-1 + 3\sqrt{3}i)(x^3y^3 + y^6)),$$

$$g(x, y) = y((-1 + 3\sqrt{3}i)(x^6 + x^3y^3) + y^6),$$

$$h(x, y) = x^6 + \left(\frac{5 - 3\sqrt{3}i}{2}\right)x^3y^3 + y^6.$$

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There is one degree 9 solution, with real integral coefficients:

$$f(x, y) = x^9 + 6x^6y^3 + 3x^3y^6 - y^9,$$

$$g(x, y) = -x^9 + 3x^6y^3 + 6x^3y^6 + y^9,$$

$$h(x, y) = 3xy(x^6 + x^3y^3 + y^6).$$

In addition to the symmetries mentioned earlier, there is a natural composition of two solutions to the Viéte equation. Suppose

$$x^3 + y^3 = p_1^3(x, y) + q_1^3(x, y) = p_2^3(x, y) + q_2^3(x, y).$$

Then if we compose the solutions, we see that

$$\begin{aligned} p_1^3(p_2(x, y), q_2(x, y)) + q_1^3(p_2(x, y), q_2(x, y)) \\ = p_2^3(x, y) + q_2^3(x, y) = x^3 + y^3. \end{aligned}$$

Accordingly, we define  $(p_1, q_1) \circ (p_2, q_2) = (p_3, q_3)$  by

$$p_3(x, y) = p_1(p_2(x, y), q_2(x, y)); q_3(x, y) = q_1(p_2(x, y), q_2(x, y)).$$

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The small solution composed with itself gives the (real) degree 9 solution: the roots of unity cancel!

Viète's solution and the small solution commute, giving the (unique) solution of degree 12, which is not written here.



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Assuming  $X_j^3 + Y_j^3 = A$ , the addition law works out to be

$$(X_1, Y_1) + (X_2, Y_2) = (X_3, Y_3),$$

where

$$X_3 = \frac{A(Y_1 - Y_2) + X_1 X_2 (X_1 Y_2 - X_2 Y_1)}{(X_1^2 X_2 + Y_1^2 Y_2) - (X_1 X_2^2 + Y_1 Y_2^2)},$$
$$Y_3 = \frac{A(X_1 - X_2) + Y_1 Y_2 (X_2 Y_1 - X_1 Y_2)}{(X_1^2 X_2 + Y_1^2 Y_2) - (X_1 X_2^2 + Y_1 Y_2^2)}.$$

But this formula breaks down when the two points coincide; instead, take a line tangent to the curve at  $(X_1, Y_1)$ . By implicit differentiation, the slope is  $-\frac{X_1^2}{Y_1^2}$  and we seek  $t$  so that

$$(X_1 - t)^3 + \left( Y_1 + t \cdot \frac{X_1^2}{Y_1^2} \right)^3 = X_1^3 + Y_1^3$$

It turns out that there is a double root at  $t = 0$  and a single root at  $t = -\frac{3X_1Y_1^3}{X_1^3 - Y_1^3}$ . Putting this value of  $t$  above gives Viète's identity.

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Let's suppose  $X, Y, A \in \mathbb{C}(t)$ , and  $A = 1 + t^3$ . Then our equation is

$$X^3(t) + Y^3(t) = 1 + t^3 \quad (9)$$

and if we homogenize (9), by setting  $t = y/x$  and multiplying both sides by  $x^3$ , then we get (6). In order to fit in this interpretation, though, keep in mind that every solution  $(p, q)$  corresponds to 18 points on the curve (9):  $(\omega^j p, \omega^k q)$  and  $(\omega^j q, \omega^k p)$ ,  $0 \leq j, k \leq 2$ .

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We recall notation and give our joint results with Rouse. Suppose

$$x^3 + y^3 = p^3(x, y) + q^3(x, y) = \left( \frac{f(x, y)}{h(x, y)} \right)^3 + \left( \frac{g(x, y)}{h(x, y)} \right)^3$$

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- Any two solutions commute under composition, up to multiplication by cube roots of unity.
- Any solution of degree  $3k$  is the composition of the small solution with a solution of degree  $k$ .
- No monomial occurring in any  $f, g, h$  has an exponent  $\equiv 2 \pmod{3}$ .

- The set of solutions form the group  $\mathbb{Z} + \mathbb{Z} + \mathbb{Z}_3$ , with generators  $(x, y)$ ,  $(\omega x, \omega y)$  and torsion involving  $\omega^j$ . The solution  $m(x, y) + n(\omega x, \omega y)$  has degree  $m^2 - mn + n^2$ .

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- The subgroup  $\mathbb{Z} + \mathbb{Z}$  is actually **ring**-isomorphic to  $\mathbb{Z}[\omega]$ , under the operations of addition of points and composition.
- Let  $a(d)$  denote the number of solutions of degree  $d$ , then

$$1 + 6 \sum_{d=1}^{\infty} a(d)x^d = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x^{m^2 - mn + n^2} \implies \text{(Lorenz, Ramanujan)}$$

$$\sum_{d=1}^{\infty} a(d)z^d = \sum_{i=0}^{\infty} \left( \frac{x^{3i+1}}{1 - x^{3i+1}} - \frac{x^{3i+2}}{1 - x^{3i+2}} \right).$$



- The set of solutions form the group  $\mathbb{Z} + \mathbb{Z} + \mathbb{Z}_3$ , with generators  $(x, y)$ ,  $(\omega x, \omega y)$  and torsion involving  $\omega^j$ . The solution  $m(x, y) + n(\omega x, \omega y)$  has degree  $m^2 - mn + n^2$ .
- The subgroup  $\mathbb{Z} + \mathbb{Z}$  is actually **ring**-isomorphic to  $\mathbb{Z}[\omega]$ , under the operations of addition of points and composition.
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- Thus,  $a(d)$  is the number of  $j \mid d$  so that  $j \equiv 1 \pmod{3}$  minus the number of  $j \equiv 2 \pmod{3}$ ; e.g.  $a(1729) = 8$ . Any degree has the form  $m^2 \prod_j p_j$ ,  $p_j \equiv 1 \pmod{3}$ .

I can't resist mentioning one more cubic identity.

$$\begin{aligned}(\beta x^2 - xy + \beta y^2)^3 + \beta(-x^2 + \beta xy - y^2)^3 = \\ (\beta^2 - 1)(\beta x^3 + y^3)(x^3 + \beta y^3)\end{aligned}$$

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It can be proved that any nontrivial sum of cubes of quadratic forms  $f_1^3 + f_2^3 + f_3^3 + f_4^3 = 0$  is, after an invertible linear change of variables, based on the equality of two of the above pairs.

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The proof requires another talk.

Recall (3), proved by Desboves (1880) and Elkies (1995): let

$$f_1(x, y) = x^2 + \sqrt{2} x y - y^2, \quad f_2(x, y) = i x^2 - \sqrt{2} x y + i y^2$$

$$f_3(x, y) = -x^2 + \sqrt{2} x y + y^2, \quad f_4(x, y) = -i x^2 - \sqrt{2} x y - i y^2$$

Then  $\sum_{i=1}^4 f_i^5 = 0$ .

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This was derived by taking the sum

$$\sum_{k=0}^3 (i^k x^2 + i^{2k} a x y + i^{3k} y^2)^5 = 40a(a^2 + 2)(x^7 y^3 + x^3 y^7),$$

and setting first  $a = \sqrt{-2}$  and then  $y \mapsto iy$ .

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The interplay of the roots of unity makes it unsurprising that

$$\sum_{i=1}^4 f_i = \sum_{i=1}^4 f_i^2 = 0$$

as well. This is actually, however, too much of a good thing.



Note that the equations  $\sum f_i = \sum f_i^2 = 0$  define the intersection of a plane and a sphere in  $\mathbb{C}^4$ . This is, projectively, a curve. Unless something special is going on, this curve shouldn't contain another curve  $(f_1, f_2, f_3, f_4)$ .

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To solve  $\sum f_i = \sum f_i^2 = 0$ , set  $f_4 = -(f_1 + f_2 + f_3)$ ; the sum of squares becomes essentially a Pythagorean triple, which we know how to parameterize:

$$f_1^2 + f_2^2 + f_3^2 + (f_1 + f_2 + f_3)^2 = 0 \implies$$

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$$\begin{aligned} f_1^2 + f_2^2 + f_3^2 + (f_1 + f_2 + f_3)^2 = 0 &\implies \\ (f_1 - f_3)^2 + 2(f_1 + f_3)^2 = -(f_1 + 2f_2 + f_3)^2 &\quad \implies \end{aligned}$$

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Solve for the  $f_i$ 's to recover the Desboves-Elkies example.

We could also try to synch a solution. Let  $\omega = \zeta_3$ .

$$f_1 = x^2 + axy + y^2$$

$$f_2 = \omega x^2 + axy + \omega^2 y^2$$

$$f_3 = \omega^2 x^2 + axy + \omega y^2$$

$$\implies f_1 + f_2 + f_3 = 3axy, \quad f_1^2 + f_2^2 + f_3^2 = 3(a^2 + 2)x^2y^2.$$

Let  $f_4 = -3axy$ ;  $3(a^2 + 2) + (3a)^2 = 6(2a^2 + 1) = 0$  implies  $a = \sqrt{-1/2}$  to give another solution. Again set  $y \rightarrow iy$ , then

$$(x^2 + \sqrt{1/2} xy - y^2)^5 + (\omega x^2 + \sqrt{1/2} xy - \omega^2 y^2)^5 + (\omega^2 x^2 + \sqrt{1/2} xy - \omega y^2)^5 + (-3\sqrt{1/2} xy)^5 = 0.$$

This is actually the same as the Desboves-Elkies (3) after a change of variables. Felix Klein smiles.

The relationship of  $\sum x_1, \sum x_1^2, \sum x_i^5$  has a context.

### Theorem

*If  $p(x_1, x_2, x_3, x_4)$  is any symmetric form of degree 5, then*  
 $p \in I = (\sum_{i=1}^4 x_i, \sum_{i=1}^4 x_i^2).$

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### Proof.

Let  $e_j$  denote the usual  $j$ -th elementary symmetric function. Since  $\sum_{i=1}^4 x_i = e_1$  and  $\sum_{i=1}^4 x_i^2 = e_1^2 - 2e_2$ ,  $I = (e_1, e_2)$ . By Newton's theorem, any symmetric quintic has the form  $c_1 e_1^5 + c_2 e_1^3 e_2 + c_3 e_1^2 e_3 + c_4 e_1 e_4 + c_5 e_1 e_2^2 + c_6 e_2 e_3$ , and so is in  $I$ .  $\square$

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The proof works because 5 cannot be written as a non-negative integer combination of 3 and 4, a case of the Frobenius problem. Let  $m$  and  $n$  be relatively prime positive integers  $> 1$  and let  $A(m, n)$  be the set of positive integers which **cannot** be written as  $am + bn$  for non-negative integers  $(a, b)$ . Sylvester showed in 1884 that  $\max A(m, n) = mn - m - n$ .

More generally, let  $M_{n,k}(x_1, \dots, x_n) = \sum_{j=1}^n x_j^k$ . A similar argument to the foregoing proves the following theorem.

### Theorem

*Suppose  $N \in A(n-1, n)$  is not expressible as  $a(n-1) + bn$ . Then*

$$\sum_{j=1}^n x_j^N \in \left( \sum_{j=1}^n x_j, \sum_{j=1}^n x_j^2, \dots, \sum_{j=1}^n x_j^{n-2} \right).$$

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Note that if  $n = 4$ , then  $A(3, 4) = \{1, 2, 5\}$ . The largest element in  $A(n-1, n)$  is  $n^2 - 3n + 1$ . Unfortunately, for  $n \geq 5$ , the intersection  $\bigcap_{r=1}^{n-2} \sum_{j=1}^n x_j^r$  has positive genus and so has no polynomial parameterization: despite the Theorem, there are no versions of Desboves-Elkies in higher degrees.

Mathematica and I spent some time searching for other “interesting” syncing identities, and found this one:

$$\sum_{k=0}^4 (\zeta_5^k x^2 + a x y + \zeta_5^{-k})^{14} =$$
$$f(a)(x^{24}y^4 + x^4y^{24}) + g(a)(x^{19}y^9 + x^9y^{19}) + h(a)x^{14}y^{14}$$

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Miraculously,  $f(i) = g(i) = 0$ , and  $h(i) = 5^7$ . It follows that if  $f_k(x, y) = \zeta_5^k x^2 + i x y + \zeta_5^{-k} y^2$  for  $0 \leq k \leq 4$  and  $f_5(x, y) = \sqrt{-5} x y$ , then

$$\sum_{j=0}^5 f_j^{14}(x, y) = 0. \quad (10)$$

By this time, you won't be surprised to hear me say that Felix Klein wouldn't have been surprised.

I don't know *why* (10) is true. Possible hint:

$$\sum_{j=0}^5 f_j^{2k}(x, y) = 0 \quad \text{for } k = 1, 2, 4, 7$$

and  $M_{6,1} = M_{6,2} = M_{6,4} = 0 \implies M_{6,7} = 0$ .

The question is: *why* do the  $f_j^{2'}$ 's lie on this intersection?

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Setting  $(x, y) = (z, 1)$ , we have the polynomial identity  $\sum_{j=0}^5 f_j^{14}(z, 1) = 0$ . Mark Green has shown that if  $r$  *entire* (let alone polynomial) functions  $\phi_j$  satisfy  $\sum_{j=1}^r \phi_j^N = 0$ , then  $N \leq r(r-2)$ ; 14 is not that much less than 24, so this might well be an extremal example.

In 1884, Felix Klein wrote a famous book on the icosahedron, and he used an idea which seems to make plausible some of these identities. He first considered the Riemann mapping, which gives a bijection between the unit sphere  $S^2 \subset \mathbb{R}^3$  and the extended complex plane:

$$(a, b, c) \in S^2 \iff \frac{a + ib}{1 - c} \in \mathbb{C}^*$$

$$u + iv \in \mathbb{C} \iff \left( \frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right) \in S^2$$

The north pole corresponds to the point at infinity.

What Klein does now is associate a point on the sphere with a linear form in  $(x, y)$  whose “root” is the image of the point:

$$(a, b, c) \in S^2 \iff x - \left( \frac{a + ib}{1 - c} \right) y, \quad , c \neq 1,$$

$$(0, 0, 1) \iff y.$$

Klein's goal was to start with a set of points of a regular polytope and take the product of the linear forms associated with its vertices. Linear changes of variable correspond to fractional linear changes in the roots:

$$p(x, y) = \mu \prod (x - \lambda_j y) \implies$$
$$p(ax + by, cx + dy) = \mu' \prod \left( x - \left( \frac{d\lambda_j - b}{a - c\lambda_j} \right) y \right).$$

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Every rotation of the sphere corresponds to a change in variables of this product, though not every change in variables gives a rotation of the sphere. Since regular polytopes have many rotational symmetries, Klein found that the resulting polynomials has many symmetries as well.

Here's what happens for the octahedron:

$$(\pm 1, 0, 0) \iff \pm 1 \iff x - y, x + y \iff x^2 - y^2$$

$$(0, \pm 1, 0) \iff \pm i \iff x - iy, x + iy \iff x^2 + y^2$$

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Taking the product, we obtain  $K(x, y) = xy(x^4 - y^4)$ . There are 24 rotational symmetries of the octahedron, and these are generated by  $K(x, iy) = -K(x, y)$  and  $K(y, x) = -K(x, y)$  (which aren't interesting) and  $K(x + y, x - y) = 8K(x, y)$  (which isn't entirely obvious).



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In the same way, with appropriate choices of the vertices, the cube is associated with  $x^8 - 14x^4y^4 + y^8$  and the icosahedron is associated with  $xy(x^{10} + 11ix^5y^5 + y^{10})$ .

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Note that

$$(a, b, c) \iff \frac{a + ib}{1 - c} := re^{i\theta};$$
$$-(a, b, c) \iff -\frac{a + ib}{1 + c} = -r^{-1}e^{i\theta}$$

and the resulting product is

$$(x - re^{i\theta}y)(x + r^{-1}e^{i\theta}y) = x^2 - (r - r^{-1})e^{i\theta}xy - e^{2i\theta}y^2.$$

After multiplying by  $e^{-i\theta}$ , these are perfect for the sort of synching we've been doing. In particular, two antipodal regular  $d$ -gons parallel to the  $xy$ -plane yield the familiar-looking set of quadratics

$$\{\zeta_d^{-j}x^2 - (r - r^{-1})xy - \zeta_d^jy^2 : 0 \leq j \leq d - 1\}.$$

Recall the octahedron:

$$(\pm 1, 0, 0) \iff \pm 1 \iff x - y, x + y \iff x^2 - y^2$$

$$(0, \pm 1, 0) \iff \pm i \iff x - iy, x + iy \iff x^2 + y^2$$

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If you start with a cube with vertices at

$$\left( \pm \sqrt{\frac{2}{3}}, 0, \pm \sqrt{\frac{1}{3}} \right), \left( 0, \pm \sqrt{\frac{2}{3}}, \pm \sqrt{\frac{1}{3}} \right)$$

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And if you rotate the cube so that vertices are at the north and south poles, then you get the alternate three-fold formulation.

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The dodecahedron gives 10 quadratics whose 14th powers are dependent. I don't know any *a priori* reason for the repeated appearance of “14”.

Here's (4) again:

$$\sum_{1 \leq i < j \leq 4} (x_i + x_j)^4 + (x_i - x_j)^4 = 6(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2.$$

This can be proved by noting that

$$(a + b)^4 + (a - b)^4 = 2a^4 + 12a^2b^2 + b^4$$

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Liouville used a version of this in 1859 to make the first advance on Waring's Problem since Lagrange's Four-Square Theorem.

### Theorem

*Every positive integer  $n$  is a sum of at most 53 4-th powers of integers.*

Proof.

Write  $n = t + 6m$ , where  $0 \leq t \leq 5$ . By Lagrange, write  $m = \sum_{i=1}^4 x_i^2$ , and then write  $x_i = \sum_{j=1}^4 y_{ij}^2$ . Then

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For example,  $1859 = 5 + 6 * 309 = 5 + 6 * (16^2 + 6^2 + 4^2 + 1^2)$  is one such representation, and after writing 16, 6, 4, 1 each as a sum of squares, one is led to

$$1859 = 6 \cdot 4^4 + 2 \cdot 3^4 + 9 \cdot 2^4 + 17 \cdot 1^4 + 19 \cdot 0^4.$$

This is not the best way to study Waring's problem, and 53 is far from optimal. (For example,  $1859 = 6^4 + 2 * 4^4 + 3 * 2^4 + 3 * 1^4$ , with 9 cubes.)

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Mathematicians in the rest of the 19th century gave similar formulas for degrees 6, 8 and 10 and then Hilbert (not for the first time) destroyed a cottage industry when he solved Waring's Problem in 1909. A key step was this non-constructive theorem:

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### Theorem (Hilbert Identities)

*For all  $n, r$ , let  $N = \binom{n+2r-1}{n-1}$ . Then there exist  $0 < \lambda_k \in \mathbb{Q}$  and  $\alpha_{kj} \in \mathbb{Z}, 1 \leq k \leq N, 1 \leq j \leq n$ , such that*

$$\sum_{k=1}^N \lambda_k (\alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n)^{2r} = (x_1^2 + \cdots + x_n^2)^r$$

The basic idea of the proof is to find the “average”  $2r$ -th power, where the coefficients range over the unit sphere  $S^{n-1}$ , by letting  $x$  be a **parameter** and computing

$$F_{2r}(S^{n-1}, \mu; x) := \int_{u \in S^{n-1}} (u_1 x_1 + \cdots + u_n x_n)^{2r} d\mu$$

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If  $a, b \in \mathbb{R}^n$  and  $\|a\| = \|b\|$ , then by the rotational invariance,  $F_{2r}(S^{n-1}, \mu; a) = F_{2r}(S^{n-1}, \mu; b)$ .

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Thus  $F_{2r}(S^{n-1}, \mu; x)$  is a function of  $\|x\|$  and since it is a form in the  $x_j$ 's of degree  $2r$ ,

$$F_{2r}(S^{n-1}, \mu; x) = c_{n,r}(x_1^2 + \cdots + x_n^2)^r$$

for some positive constant  $c_{n,r}$ . This constant can be computed by choosing  $x$  to be a unit vector and doing the integral.



The next step is approximate the integral with a Riemann sum and use Carathéodory's Theorem to show that each such sum can be replaced by one with at most  $N$  terms. Ultimately, an application Bolzano-Weierstrass gives a convergent subsequence. The argument that the coefficients are rational is subtle!

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It is sometimes convenient to ignore the algebraic constraints, and absorb the  $\lambda_k$ 's into the powers by writing

$$(\beta_{k1}x_1 + \cdots + \beta_{kn}x_n)^{2r} = \lambda_k(\alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n)^{2r}.$$

The rest of the talk will give some applications.

Suppose

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Dvoretzky's Theorem in functional analysis says that any infinite-dimensional Banach space contains isometric copies of every  $\ell_2$ . Hilbert Identities can be used for concrete finite-dimensional examples. Bounds on the length of a Hilbert identity correspond to bounds on the dimensions of the corresponding spaces.

For example, consider the vectors  $u_j = (\beta_{1j}, \dots, \beta_{Nj}) \in \mathbb{R}^N$ ,  $1 \leq j \leq n$ . For any  $x \in \mathbb{R}^n$ ,  $\|\sum_j x_j u_j\|_{2r}^{2r}$  is the left side, which by the right side is  $\|x\|_2^{2r}$ . In other words, the  $n$ -dimensional subspace  $\langle u_j \rangle \subset \ell_{2r}^N$  is isometric to  $\ell_2^n$ .

Suppose a set  $S$  and non-negative measure  $\mu$  are given. An *exact quadrature formula for  $(S, \mu)$  of degree  $d$*  is an expression

$$\int_{u \in S} p(u) d\mu = \sum_{k=1}^N \lambda_k p(\alpha_k),$$

which holds for **all** forms  $p$  of degree  $d$ . (Applications require  $\alpha_k \in S$  and  $\lambda_k \geq 0$ , but this is not formally necessary.)

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which holds for **all** forms  $p$  of degree  $d$ . (Applications require  $\alpha_k \in S$  and  $\lambda_k \geq 0$ , but this is not formally necessary.)

Such an equation holds if and only if it holds for all monomials:  $x^i = x_1^{i_1} \cdots x_n^{i_n}$  of degree  $d$ . Taken on the right hand side, we get the monomials in a sum of  $d$ -th powers of linear forms. It's getting kind of late in the talk, so I'll skip the derivation and get to the punch-line. I hope you trust me with the constants.

## Theorem

Suppose  $\mu$  is the rotation-invariant unit measure on  $S^{n-1}$  and  $\lambda_k \in \mathbb{R}$ ,  $\alpha_k \in \mathbb{R}^n$  and  $d \in \mathbb{N}$ . Then

$$\int_{u \in S^{n-1}} p(u) d\mu = \sum_{k=1}^N \lambda_k p(\alpha_k),$$

is an exact quadrature formula of degree  $d$  for  $(S^{n-1}, \mu)$  iff

$$\sum_{k=1}^N \lambda_k (\alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n)^d = c_{n,2d}(x_1^2 + \cdots + x_n^2)^{d/2},$$

where  $c_{n,2r} = \prod_{j=1}^r \frac{n+2j}{1+2j}$  and  $c_{n,2r+1} = 0$ . (That is, if  $d$  is odd, ignore  $d/2$ ; the sum is simply 0.)

If  $q$  is a form of degree  $d - 2i$ , then  $(\sum x_j^2)^i q$  is a form of degree  $d$  which agrees with  $q$  on  $S^{n-1}$ , so an exact quadrature formula of degree  $d$  is also one of degree  $d - 2i$ . If  $d$  is odd, the integral vanishes. By writing  $f$  as a sum of homogeneous pieces, we get



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### Corollary

If

$$\int_{u \in S^{n-1}} p(u) d\mu = \sum_{k=1}^N \lambda_k p(\alpha_k),$$

is an exact quadrature formula of degree  $d$ , then for **every** polynomial  $f$  (homogeneous or not) of degree  $\leq 2\lfloor \frac{d}{2} \rfloor + 1$ ,

$$\int_{u \in S^{n-1}} f(u) d\mu = \sum_{k=1}^N \frac{\lambda_k}{2} (f(\alpha_k) + f(-\alpha_k))$$

These establish the centrality of Hilbert Identities for quadrature formulas on  $S^{n-1}$ . Another corollary uses an old trick method from numerical analysis.

### Corollary

*In any Hilbert Identity,  $N \geq \binom{n+r-1}{n-1}$ .*

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### Corollary

*In any Hilbert Identity,  $N \geq \binom{n+r-1}{n-1}$ .*

### Proof.

If  $N < \binom{n+r-1}{n-1}$ , then there exists a non-zero form  $h$  of degree  $r$  so that  $h(\alpha_k) = 0$ ,  $1 \leq k \leq N$ . (Count the number of monomials.)  
Now put  $p = h^2$  into the quadrature formula; we have

$$\int_{u \in S^{n-1}} h^2(u) d\mu = \sum_{k=1}^N \lambda_k h^2(\alpha_k),$$

which is  $> 0$  on the left, and 0 on the right. Contradiction! □

How good an estimate is this? For  $r = 2$  and  $n = 4$ ,  $\binom{2+4-1}{4-1} = 10$ . Liouville's (4) has 12 terms. A while back, I proved that 10 is impossible, but 11 is possible:

$$\begin{aligned}
 & 12(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2 \\
 &= 6(x_1 + x_2 + x_3 + x_4)^4 + \sum_{i=1}^4 (x_2 \pm x_3 \pm x_4)^4 + \\
 & \sum_{i=1}^2 (x_1 \pm \sqrt{2}x_2)^4 + \sum_{i=1}^2 (x_1 \pm \sqrt{2}x_3)^4 + \sum_{i=1}^2 (x_1 \pm \sqrt{2}x_4)^4.
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The right-hand side is symmetric in  $\{x_2, x_3, x_4\}$ , but not in  $x_1$ .

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The right-hand side is symmetric in  $\{x_2, x_3, x_4\}$ , but not in  $x_1$ . I suspect this solution is unique, up to orthogonal changes of variable, but have been unable to prove it, in efforts spanning four different decades.

If a Hilbert Identity has minimal length, then the summands have some special properties

### Corollary

If

$$\sum_{k=1}^N (\beta_{k1}x_1 + \cdots + \beta_{kn}x_n)^{2r} = (x_1^2 + \cdots + x_n^2)^r.$$

and  $N = \binom{n+r-1}{n-1}$  is minimal, then for all  $k$ ,

$$\left( \sum_{\ell=1}^n \beta_{k\ell}^2 \right)^r = \frac{1}{N} \prod_{j=1}^r \frac{n+2j}{1+2j}.$$

This leads to the final interpretation of Hilbert Identities. In a beautiful series of papers in the 1970s, Delsarte, Goethals and Seidel introduced the idea of the spherical design.

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A set  $X = \{v_1, \dots, v_N\} \in \mathbb{R}^n$  is a *spherical  $t$ -design* if for every polynomial  $p(x_1, \dots, x_n)$ ,  $\deg p \leq t$ , we have

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There are some wonderful theorems about spherical designs.

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- If  $t = 2s$ , then  $N \geq \binom{n+s-1}{n-1} + \binom{n+s-2}{n-1}$ ; if  $t = 2s + 1$ , then  $N \geq 2\binom{n+s-1}{n-1}$ , and there exists  $N(n, t)$  so that for all  $N \geq N(n, t)$ , such a  $t$ -design with  $N$  points exists (Seymour and Zaslavsky).

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- Your favorite symmetric pointset in  $\mathbb{R}^n$  is a spherical design.
- A tight spherical  $2s + 1$ -design in  $\mathbb{R}^n$  defines the maximal number of lines through the origin in  $\mathbb{R}^n$  which make only  $s$  different angles with each other.



- Tight spherical  $2s + 1$ -designs exist whenever  $n = 2$  and  $2s + 1 = 3$  and for  $(2s + 1, n) = (5,7), (5,23), (7,8), (7,23), (11,24)$ . Otherwise, they are impossible unless  $2s + 1 = 5$  and  $n = u^2 - 2$  ( $u$  odd) or  $2s + 1 = 7$  and  $n = 3v^2 - 4$ . Some non-existence results exist, but many cases remain open.

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- No new tight spherical designs have been found in the last 30 years. All known tight spherical designs are unique, up to rotation. All known proofs of this are *ad hoc*.
- Tight spherical designs lead to beautiful Hilbert Identities, as in (4). Take the indices below as cyclic mod 7, then

$$\sum_{i=1}^7 \sum_{\pm} (x_i \pm x_{i+1} \pm x_{i+3})^4 = 12(x_1^2 + \cdots + x_7^2)^2.$$

This comes from the finite projective plane of order 2.

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- The tight 11-design in  $\mathbb{R}^{24}$  is derived from the minimal vectors in the Leech lattice and has the following hilarious implication:
- There is an isometric copy of  $\ell_2^{24}$  in  $\ell_{10}^{98280}$ , but not in  $\ell_{10}^{98279}$ .
- Using the Schönemann coordinates for an icosahedron and letting  $\Phi = \frac{\sqrt{5}+1}{2}$ , so that  $\Phi^4 + 1 = 3\Phi^2$ , we have

$$6\Phi^2(x^2 + y^2 + z^2)^2 = (\Phi x + y)^4 + (\Phi x - y)^4 + (\Phi y + z)^4 + (\Phi y - z)^4 + (\Phi z + x)^4 + (\Phi z - x)^4.$$

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### Theorem

*If the equation*

$$(x_1^2 + x_2^2 + x_3^2)^2 = \sum_{k=1}^r (a_k x_1 + b_k x_2 + c_k x_3)^4 \quad (11)$$

*holds, then  $r \geq 6$ . If  $r = 6$ , then this equation is true if and only if the 12 points  $\pm(a_k, b_k, c_k)$  are the vertices of a regular icosahedron inscribed in a sphere with center 0 and radius  $(5/6)^{1/4}$ .*



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### Theorem

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$$(x_1^2 + x_2^2 + x_3^2)^2 = \sum_{k=1}^r (a_k x_1 + b_k x_2 + c_k x_3)^4 \quad (12)$$

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