Steampunk canonical forms, or Recent results in 19th century algebra

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Algebraic Geometry Seminar
University of Chicago, December 8, 2011

1. Introduction

Much of this material will appear in the paper *Steampunk canonical forms*, which I hope to submit by the end of the year. I'll put these slides on my webpage:
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Many thanks to the Lek-Heng Lim for the invitation to visit the campus and to Madhav Nori for the invitation to speak in this seminar.

2. Overview

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You don't need a proof of this, but I'll give you several anyway.

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Note that there were three parameters in the two representations of binary quadratic forms on the last slide. This is necessary, but not sufficient. Obviously, $\sum_{i=1}^{3}(t_i x)^2$ isn't a canonical form. More subtly, there are three parameters in

$$(t_1 x + t_2 y)^2 + (i t_1 x + t_3 y)^2,$$

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This example may look silly, but it isn't.

3. Well-known old canonical forms

There are two sets of well-known 19th century canonical forms; both could be construed as generalizations of the Main Example.

Every quadratic form $p \in H_2(\mathbb{C}^n)$ is a sum of n squares of linear forms, but since the naive number of coefficients in such a sum, $n \times n$, is greater than $N(n,2) = \frac{n(n+1)}{2}$, a sum of n squares is not, *per se*, a canonical form.

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$$F(t,x) := \sum_{i=1}^{n} (t_{ii}x_i + \cdots + t_{in}x_n)^2.$$

Here, $\{t_{ij} : 1 \le i \le j \le n\}$ are the $N(n,2)$ parameters. The usual proof of this is constructive: just iteratively complete the square.

3. Well-known old canonical forms

In 1851, J. J. Sylvester proved a once-famous theorem about canonical forms for binary forms, using a still-handy algorithm.

Theorem (Sylvester's Canonical Forms)

(i) A general binary form of odd degree $d = 2k - 1$ can be written uniquely (i.e., up to indexing and roots of unity) as

$$\sum_{j=1}^{k} (\alpha_j x + \beta_j y)^{2k-1}.$$

(ii) A general binary form of even degree $d = 2k$ can be written uniquely as

$$\lambda \cdot x^{2k} + \sum_{j=1}^{k} (\alpha_j x + \beta_j y)^{2k}.$$

for some $\lambda \in \mathbb{C}$.

4. Some new steampunk canonical forms

Here are a few examples of what I'll be talking about and proving (the non-constructive proofs are quite simple):

Theorem

A general binary sextic form is the sum of the cube of a quadratic form and the square of a cubic form.

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Theorem

A general binary form of degree $d = 2k$ can be written as

$$(\lambda_0\, x^2 + \lambda_1\, x\, y + \lambda_2\, y^2)^k + \sum_{j=1}^{k-1} (\alpha_j x + \beta_j y)^{2k}.$$

4. Some new steampunk canonical forms

Theorem (Slinky)

A general cubic form $p(x_1, \dots, x_n)$ *has a unique representation as*

$$p(x_1, \dots, x_n) = \sum_{1 \le i \le j \le n} (\alpha_{\{i,j\},i} x_i + \cdots + \alpha_{\{i,j\},j} x_j)^3.$$

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Boris Reichstein found a wonderful and little-known canonical form in 1987, which I'll discuss later.

Theorem (Reichstein's Theorem)

A general cubic $p(x_1, \dots, x_n)$ can be written as

$$\sum_{k=1}^{n} (\alpha_{k1} x_1 + \cdots + \alpha_{kn} x_n)^3 + q(x_1, \dots, x_{n-2}).$$

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It is well-known that a general binary quartic form, after a change of variables, can be written as $x^4 + 6\lambda x^2 y^2 + y^4$. It is easy to prove that this extends to higher degrees:

Theorem

After a linear change of variables, a general $p \in H_d(\mathbb{C}^2)$ can be written as

$$p(x,y) = x^d + \sum_{k=2}^{d-2} \binom{d}{k} t_k x^{d-k} y^k + y^d.$$

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But $15 = 3 \times 3 + 1 \times 6$: a general ternary quartic *is* a sum of three fourth powers of linear forms plus the square of a quadratic form.

5. Basic Definitions

Let $H_d(\mathbb{C}^n)$ denote the set of forms $p(x_1, \dots, x_n)$ of degree d with coefficients in $\mathbb{C}$. The dimension of the vector space $H_d(\mathbb{C}^n)$ is $N(n,d) := \binom{n+d-1}{d}$. Let $\mathcal{I}(n,d)$ be the index set:

$$\mathcal{I}(n,d) = \left\{ (i_1, \dots, i_n) : 0 \le i_k \in \mathbb{Z}, \quad \sum_k i_k = d \right\}.$$

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Let $x^i = x_1^{i_1} \cdots x_n^{i_n}$ and $c(i) = \frac{d!}{\prod i_k!}$ denote the multinomial coefficient. If $p \in H_d(\mathbb{C}^n)$, then we can write

$$p(x_1, \dots, x_n) = \sum_{i \in \mathcal{I}(n,d)} c(i) a(p;i) x^i.$$

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Theorem

Suppose $F : \mathbb{C}^N \to \mathbb{C}^N$ is a polynomial map; that is,

$$F(t_1, \dots, t_N) = (f_1(t_1, \dots, t_N), \dots, f_N(t_1, \dots, t_N))$$

where each $f_j \in \mathbb{C}[t_1, \dots, t_N]$. Then either (i) or (ii) holds:
(i) The N polynomials $\{f_j : 1 \le j \le N\}$ are algebraically dependent and $F(\mathbb{C}^N)$ lies in some non-trivial $\{P = 0\}$ in $\mathbb{C}^N$.
(ii) The N polynomials $\{f_j : 1 \le j \le N\}$ are algebraically independent and $F(\mathbb{C}^N)$ is almost all of $\mathbb{C}^N$.
Furthermore, the second case occurs if and only there is a point $u \in \mathbb{C}^N$ at which the Jacobian matrix $\left[\frac{\partial f_i}{\partial t_j}(u)\right]$ has full rank.

5. Basic Definitions

When $N = N(n,d)$, we may interpret such an F as a map from $\mathbb{C}^N$ to $H_d(\mathbb{C}^n)$ by indexing $\mathcal{I}(n,d)$ as $\{i_j : 1 \le j \le N\}$ and making the interpretation in an abuse of notation that

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Definition

A **canonical form** for $H_d(\mathbb{C}^n)$ is any polynomial map F from $\mathbb{C}^N$ to $H_d(\mathbb{C}^n)$ so that almost every $p \in H_d(\mathbb{C}^n)$ is in the range of F.

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If $\{\phi_j\}$ is a basis of $H_d(\mathbb{C}^n)$, e.g. $\phi_j(x) = c(i_j)x^{i_j}$, then

$$F(t,x) = \sum_{j=1}^{N} t_j\phi_j(x)$$

is technically (though not traditionally) a canonical form.

5. Basic Definitions

One gets the impression from the literature that canonical forms are extremely rare. Actually, a "general" polynomial map from $\mathbb{C}^N$ to $H_d(\mathbb{C}^n)$ is canonical. But *interesting* canonical forms, those with natural interpretations, are rare enough to be noteworthy.

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Let's return to the Main Example.
The partials of $(t_1x_1 + t_2x_2)^2 + (t_3x_2)^2$ with respect to the t_j's are:

$$2x_1(t_1x_1 + t_2x_2), \quad 2x_2(t_1x_1 + t_2x_2), \quad 2x_2(t_3x_2).$$

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If we specialize at $(t_1, t_2, t_3) = (1, 0, 1)$, so $t_1x_1 + t_2x_2 = x_1$ and $t_3x_2 = x_2$, then these partials become $2x_1^2, 2x_1x_2, 2x_2^2$. Since these span $H_2(\mathbb{C}^2)$, we get an abstract existential proof that you can complete the square for binary quadratic forms!

5. Basic Definitions

We can prove the sextic theorem in a similar fashion. Suppose

$$p(x,y) = f^2(x,y) + g^3(x,y) =$$

$$(t_1x^3 + t_2x^2y + t_3xy^2 + t_4y^3)^2 + (t_5x^2 + t_6xy + t_7y^2)^3 :$$

$$f(x,y) = t_1x^3 + t_2x^2y + t_3xy^2 + t_4y^3,$$

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Then the partials with respect to the t_j's are:

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Once again, this is obviously a basis for $H_6(\mathbb{C}^2)$. I haven't been able to find a constructive proof of this theorem yet.

5. Basic Definitions

It is well-known that for a general binary binary quartic, there is an invertible linear change of variables after which $x^4 + 6\lambda x^2y^2 + y^4$. (In fact each quartic corresponds to six such values of λ.) A generalization of this can be easily proved.

Theorem

After a linear change of variables, a general $p \in H_d(\mathbb{C}^2)$ can be written as

$$p(x,y) = x^d + \sum_{k=2}^{d-2} \binom{d}{k} t_k x^{d-k} y^k + y^d.$$

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To sketch the proof, let $W = \alpha_1 x + \alpha_2 y, Z = \alpha_3 x + \alpha_4 y$. Take the partials with respect to the α_j's and t_k's and then set $W = x, Z = y$ and $t_k = 0$.

5. Basic Definitions

By looking at the coefficients of $p(x,y) = \prod(x - z_k y)$, we immediately obtain the following corollary.

Corollary

For a general set of n complex numbers $z_k \in \mathbb{C}$, there exists a Möbius transformation T so that

$$\sum_{k=1}^{n} T(z_k) = \sum_{k=1}^{n} T(\tfrac{1}{z_k}) = 0.$$

With a bit more work, the missing monomials $W^{d-1}Z, WZ^{d-1}$ on the last slide may be replaced by any two monomials, except $\{W^d, W^{d-1}Z\}$ and $\{WZ^{d-1}, Z^d\}$, whose absence would imply a square factor in p. I don't know how many different choices of $\{t_k\}$ occur.

5. Basic Definitions

There are two ways to show that F is a canonical form. One way is to use the Theorem and find a single point u at which the Jacobian has full rank, that is, where $\left\{\frac{\partial F}{\partial t_j}(u)\right\}$ spans $H_d(\mathbb{C}^n)$. This reduces verification to an REU topic, I say this with great affection and respect for undergraduate research.

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This has often be done via apolarity, see below. With the apolarity interpretation, this is known classically as the Lasker-Wakeford Theorem. A beautiful modern version is given in Ehrenborg-Rota.

The second (and better way, if possible) is to give a constructive algorithm for writing a general form in $H_d(\mathbb{C}^n)$ in the shape $F(t; x)$. These are usually *ad hoc*.

6. Apolarity

The following bilinear form can be found in ancient invariant theory; analysts call it the "Fisher inner product". It's at least 120 years old.

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Recall that

$$p(x_1 \dots, x_n) = \sum_{i \in \mathcal{I}(n,d)} c(i) a(p;i) x^i.$$

For $p, q \in H_d(\mathbb{C}^n)$, let

$$[p, q] = \sum_{i \in \mathcal{I}(n,d)} c(i) a(p;i) a(q;i).$$

(Ehrenborg-Rota uses this extensively, but never explicitly.)

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(Ehrenborg-Rota uses this extensively, but never explicitly.)
This is only an inner product for real forms; for complex forms you need $\overline{a(q;i)}$. The conjugate actually only makes our expressions more complicated, $[p,q]$ is really just a bilinear form on $H_d(\mathbb{C}^n)$.

6. Apolarity

Definition

$p, q \in H_d(\mathbb{C}^n)$ are **apolar** if $[p, q] = 0$.

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For $\alpha \in \mathbb{C}^n$, define $(\alpha\cdot)^d \in H_d(\mathbb{C}^n)$ by

$$(\alpha\cdot)^d(x) = (\alpha \cdot x)^d = \left(\sum_{j=1}^{n} \alpha_j x_j\right)^d = \sum_{i \in \mathcal{I}(n,d)} c(i)\alpha^i x^i,$$

where the usual multinomial conventions apply. We define the differential operator $q(D)$ for $q \in H_e(\mathbb{C}^n)$ in the usual way by

$$q(D) = \sum_{i \in \mathcal{I}(n,e)} c(i)a(q;i)\left(\frac{\partial}{\partial x_1}\right)^{i_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{i_n}.$$

The reason $[p, q]$ is so useful is that it has so many nice properties; all can be verified formally.

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- If $e \le d$ and $g \in H_{d-e}(\mathbb{C}^n)$, then

$$g(D)(\alpha\cdot)^d = \frac{d!}{e!}g(\alpha)(\alpha\cdot)^e.$$

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- If $\deg q \le \deg p$ and $q(D)p = 0$, then all multiples of q in $H_d(\mathbb{C}^n)$ are apolar to p.
- Classically, if p and q are forms of possibly different degree, p is apolar to q if $p(D)q = 0$. The definitions coincide when the degrees are equal, but not otherwise. In that case, the definition is not symmetric: if $\deg p > \deg q$, then $p(D)q$ will always equal 0.

7. Why doesn't constant-counting work?

Why are canonical forms even an issue? The main reason is that maps which one would think have full range don't. Apart from sums of squares, where the orthogonal group plays a role, the simplest example occurs in $H_4(\mathbb{C}^3)$. Since $N(3,4) = \binom{6}{2} = 15$, one expects that a general ternary quartic could be written as

$$p(x_1, x_2, x_3) = \sum_{k=1}^{5} (\alpha_{k1}x_1 + \alpha_{k2}x_2 + \alpha_{k3}x_3)^4$$

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This would be a canonical form, if the partials with respect to the α_{kj}'s at some chosen value would span $H_4(\mathbb{C}^3)$. By apolarity, this means that there should be no non-zero quartic which is singular at the five points $\alpha_k = (\alpha_{k1}, \alpha_{k2}, \alpha_{k3})$.

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However, as Clebsch argued in the 1860's, since $N(3,2) = 6$, any choice of five α_k's pass through a non-zero quadratic $h(x_1, x_2, x_3)$, and so h^2 will be apolar to all the partials and a sum of five 4th powers is not a canonical form.

7. Why doesn't constant-counting work?

A few years later, Sylvester gave another proof. Given

$$p(x_1, x_2, x_3) = \sum_{r+s+t=4} \frac{4!}{r!s!t!} a_{rst} x_1^r x_2^s x_3^t,$$

define the catalecticant H_p as a quadratic form in 6 variables (or a 6×6 symmetric matrix defined linearly in terms of p).

$$H_p = \begin{pmatrix} a_{400} & a_{220} & a_{202} & a_{310} & a_{301} & a_{211} \\ a_{220} & a_{040} & a_{022} & a_{130} & a_{121} & a_{031} \\ a_{202} & a_{022} & a_{004} & a_{112} & a_{103} & a_{013} \\ a_{310} & a_{130} & a_{112} & a_{220} & a_{211} & a_{121} \\ a_{301} & a_{121} & a_{103} & a_{211} & a_{202} & a_{112} \\ a_{211} & a_{031} & a_{013} & a_{121} & a_{112} & a_{022} \end{pmatrix}$$

7. Why doesn't constant-counting work?

Under this definition, $H_{(\alpha\cdot)^4}$ is a perfect square. Thus if p is a sum of five fourth powers, then $rank(H_p) \le 5$, so H_p is singular. This can't happen for a general ternary quartic, where the determinant is non-zero. This gives the algebraic relation of the coefficients in Clebsch's proof.

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Clebsch's proof and Sylvester's proof are really the same, because as a quadratic form,

$$H_p(t_1, \dots, t_6) =$$
$$[(t_1x_1^2 + t_2x_2^2 + t_3x_3^2 + t_4x_1x_2 + t_5x_1x_3 + t_6x_2x_3)^2(D)]p.$$

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Our 19th century ancestors saw that funny things happen when $(n, d) = (3, 4), (4, 4), (5, 4), (5, 3)$. In the early 1990s, Alexander and Hirschowitz proved that these are the only cases in which this can happen.

7. Why doesn't constant-counting work?

But wait a minute. $N(3,4) = 15 = 5 \times 3$ to be sure, but it's also equal to $3 \times 3 + 1 \times 6$, so conceivably

$$p(x_1, x_2, x_3) = \sum_{k=1}^{3} (\alpha_{k1}x_1 + \alpha_{k2}x_2 + \alpha_{k3}x_3)^4 + (q(x_1, x_2, x_3))^2,$$

where $q \in H_2(\mathbb{C}^3)$, might be a canonical form for $H_4(\mathbb{C}^3)$.

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where $q \in H_2(\mathbb{C}^3)$, might be a canonical form for $H_4(\mathbb{C}^3)$.

Theorem

It is!

Sketch of Proof: Evaluate the Jacobian where the three linear forms are specialized to x, y, z and the quadratic is specialized to $xy + xz + yz$.

8. The Lasker-Wakeford Theorem

And now, a biographical interlude.

Emanuel Lasker (1868-1941) received his Ph.D. under Max Noether at Göttingen in 1902. He first developed the concept of a primary ideal and proved the primary decomposition theorem for an ideal of a polynomial ring in terms of primary ideals.

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If you remember European history, those dates will give you pause.

8. The Lasker-Wakeford Theorem

The memorial article by J. H. Grace about Wakeford in the *Proceedings of the London Mathematical Society* may be the angriest obituary I've ever read in a scholarly journal, and it can be found in its entirety on my webpage:

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"He [EKW] was slightly wounded early in 1916, and soon after coming home was busy again with Canonical Forms.... [H]e discovered a paper of Hilbert's which contained the very theorem he had long been in want of – first vaguely, and later quite definitely. This was in March; April found him, full of the most joyous and reverential admiration for the great German master, working away in fearful haste to finish the dissertation ...

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He returned to the front in June and was killed in July.... He only needed a chance, and he never got it."

8. The Lasker-Wakeford Theorem

This attribution comes from *The theory of determinants, matrices and invariants* by H. W. Turnbull, who was one of the last "pre-Hilbert" invariant theorists. This 1960 book is a Rosetta Stone for understanding 19th century algebra. Turnbull described this theorem as "paradoxical and very curious".

Theorem (Lasker-Wakeford)

If $F : \mathbb{C}^N \to H_d(\mathbb{C}^n)$, then F is a canonical form if and only if there is a point $u \in \mathbb{C}^N$ so that there is no non-zero form q which is apolar to all N forms $\{\frac{\partial F}{\partial t_1}(u), \dots, \frac{\partial F}{\partial t_N}(u)\}$.

The point is simply that the set $\{\frac{\partial F}{\partial t_1}(u), \dots, \frac{\partial F}{\partial t_N}(u)\}$ spans $H_d(\mathbb{C}^n)$ if and only if its perp is $\{0\}$.

8. The Lasker-Wakeford Theorem

An appeal to apolarity is often unnecessary. For binary forms, zeros correspond to linear factors and counting is all we need to do. The classical "Fundamental Theorem of Apolarity" can now be easily stated and proved. I don't know how it was understood before the Nullstellensatz. The theorem applies even when $e > d$. More modern versions have been studied by Helgason.

Theorem (FTA)

Suppose $q \in H_e(\mathbb{C}^n)$ is irreducible and $p \in H_d(\mathbb{C}^n)$. Then $q(D)p = 0$ iff there exist $\alpha_k \subset \{\alpha : q(\alpha) = 0\}$ and $\lambda_k \in \mathbb{C}$ such that

$$p(x) = \sum_{k=1}^{m} \lambda_k (\alpha_k \cdot x)^d.$$

Proof.

Fix q. Define the two subspaces

$$A = \{p \in H_d(\mathbb{C}^n) : q(D)p = 0\},$$
$$B = \{p \in H_d(\mathbb{C}^n) : p = \sum \lambda_k(\alpha_k \cdot)^d, \quad q(\alpha_k) = 0\}.$$

We want to show that $B \subseteq A$ and $B^\perp \subseteq A^\perp$; if so, then since $H_d(\mathbb{C}^n)$ is finite dimensional, it will follow that $A = B$.

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First, if $e > d$, then $A = H_d(\mathbb{C}^n)$, so $B \subseteq A$. If $e \le d$ and $p \in B$, then $q(D)p = \frac{d!}{(d-e)!} \sum \lambda_k q(\alpha_k)(\alpha_k \cdot)^{d-e} = 0$, so $p \in A$.

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Observe that $f \in B^\perp$ iff $q(\alpha) = 0$ implies $[f, (\alpha \cdot)^d] = f(\alpha) = 0$. Since q is irreducible, the Nullstellensatz implies that $q \mid f$. If $e > d$, this is impossible unless $f = 0$, so $B^\perp = \{0\} \subseteq A^\perp$. If $e \le d$, then $f = gq$ where $g \in H_{d-e}(\mathbb{C}^n)$. But $p \in A \implies q(D)p = 0 \implies [p, f] = [p, gq] = \frac{d!}{(d-e)!}[q(D)p, g] = 0$. It follows that $f \in A^\perp$, completing the proof. □

8. The Lasker-Wakeford Theorem

About 15 years ago, I proved a fairly obvious generalization of this theorem when q is not irreducible. The proof is, in spirit, the same as the one given above.

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Theorem

Suppose $q \in H_e(\mathbb{C}^n)$ factors as $\prod_{j=1}^r q_j^{m_j}$ into a product of distinct irreducible factors and suppose $p \in H_d(\mathbb{C}^n)$. Then $q(D)p = 0$ iff there exist $\alpha_{jk} \subset \{q_j(\alpha) = 0\}$, and $\phi_{jk} \in H_{m_j-1}(\mathbb{C}^n)$ such that

$$p(x) = \sum_{j=1}^{r} \left(\sum_{k=1}^{n_j} \phi_{jk}(x)(\alpha_{kj} \cdot x)^{d-(m_j-1)} \right).$$

The *Mathematical Reviews* comment on this paper included the sentence: "The proof is remarkably elementary." That's my all-time favorite review!

8. The Lasker-Wakeford Theorem

In the case of binary forms, the FTA is even simpler, and can be made equivalent to Gundelfinger's generalization of Sylvester's canonical forms. This next theorem appears in Ehrenborg-Rota.

Theorem

Suppose $\sum_{j=1}^{r} m_j = d+1$ and let $\ell_i(x,y) = \alpha_i x + \beta_i y$. Suppose further that ℓ_k and ℓ_ℓ are pairwise linearly independent for $k \neq \ell$. Then the following set is a basis for $H_d(\mathbb{C}^2)$:

$$\left\{ x^{(m_j-1)-k} y^k (\beta_j x - \alpha_j y)^{d-(m_j-1)} : 0 \le k \le m_j - 1, 1 \le j \le r \right\}$$

8. The Lasker-Wakeford Theorem

Remember canonical forms? Suppose $t_1, \dots, t_n$ appear in a canonical form as

$$(t_1x_1 + \cdots + t_nx_n)^d = \ell^d.$$

Then $\frac{\partial F}{\partial t_j} = dx_j\ell^{d-1}$, and in applying Lasker-Wakeford, note that a form is apolar to each of these if and only if it is singular at $(t_1, \dots, t_n)$. Start thinking about general forms which are singular at general sets of points and you enter the context in which Alexander-Hirschowitz comes in.

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The rest of this talk will consider first new canonical forms for binary forms, and then for cubic forms.

9. Sylvester's canonical Forms

In 1869, Sylvester (1814-1897) reflected on the discovery of his canonical forms, done while he was working as an actuary in the office next to Cayley's.

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"I discovered and developed the whole theory of canonical binary forms for odd degrees, and, as far as yet made out, for even degrees too, at one evening sitting, with a decanter of port wine to sustain nature's flagging energies, in a back office in Lincoln's Inn Fields. The work was done, and well done, but at the usual cost of racking thought — a brain on fire, and feet feeling, or feelingless, as if plunged in an ice-pail. That night we slept no more."

9. Sylvester's canonical Forms

In 1869, Sylvester (1814-1897) reflected on the discovery of his canonical forms, done while he was working as an actuary in the office next to Cayley's.

"I discovered and developed the whole theory of canonical binary forms for odd degrees, and, as far as yet made out, for even degrees too, at one evening sitting, with a decanter of port wine to sustain nature's flagging energies, in a back office in Lincoln's Inn Fields. The work was done, and well done, but at the usual cost of racking thought — a brain on fire, and feet feeling, or feelingless, as if plunged in an ice-pail. That night we slept no more."

The "λx^{2k}" must be what Sylvester meant by "as far as yet made out". As if anticipating modern mathematical preferences, Sylvester proved his theory with one brilliant algorithm.

9. Sylvester's canonical forms

Theorem (Sylvester)

Suppose $p(x,y) = \sum_{j=0}^{d} \binom{d}{j} a_j x^{d-j} y^j$ *and* $\{\alpha_k x + \beta_k y\}$ *is a set of pairwise distinct linear factors. Let* $h(x,y) = \sum_{t=0}^{r} c_t x^{r-t} y^t$ $= \prod_{j=1}^{r} (\beta_j x - \alpha_j y)$*. Then there exist* $\lambda_k \in \mathbb{C}$ *so that*

$$p(x,y) = \sum_{k=1}^{r} \lambda_k (\alpha_k x + \beta_k y)^d$$

if and only if

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_r \\ a_1 & a_2 & \cdots & a_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d-r} & a_{d-r+1} & \cdots & a_d \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

9. Sylvester's canonical forms

Here is an example of Sylvester's algorithm in action. Let

$$p(x,y) = 3x^5 - 20x^3y^2 + 10xy^4 =$$

$$\binom{5}{0}\cdot 3\ x^5 + \binom{5}{1}\cdot 0\ x^4y + \binom{5}{2}\cdot(-2)\ x^3y^2$$

$$+\binom{5}{3}\cdot 0\ x^2y^3 + \binom{5}{4}\cdot 2\ xy^4 + \binom{5}{5}\cdot 0\ y^5;$$

$$\begin{pmatrix} 3 & 0 & -2 & 0 \\ 0 & -2 & 0 & 2 \\ -2 & 0 & 2 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We have $h(x,y) = y(x^2+y^2) = y(y-ix)(y+ix)$.

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We have $h(x,y) = y(x^2+y^2) = y(y-ix)(y+ix)$.
Accordingly, there exist $\lambda_k \in \mathbb{C}$ so that

$$p(x,y) = \lambda_1 x^5 + \lambda_2(x+iy)^5 + \lambda_3(x-iy)^5.$$

Indeed, $\lambda_1 = \lambda_2 = \lambda_3 = 1$, as may be checked.

9. Sylvester's canonical forms

A few remarks about Sylvester's algorithm

- If $h(D) = \prod_{j=1}^{r}(\beta_j \frac{\partial}{\partial x} - \alpha_j \frac{\partial}{\partial y}) = \sum_{t=0}^{r} c_t \frac{\partial^r}{\partial x^{r-t} \partial y^t}$, then

$$h(D)p = \sum_{m=0}^{d-r} \frac{d!}{(d-r-m)!m!} \left(\sum_{i=0}^{d-r} a_{i+m} c_i \right) x^{d-r-m} y^m$$

  The coefficients of $h(D)p$ are, up to multiple, the rows in the matrix product, so the matrix condition is $h(D)p = 0$. The algorithm is the FTA configured for products of linear factors.

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The coefficients of $h(D)p$ are, up to multiple, the rows in the matrix product, so the matrix condition is $h(D)p = 0$. The algorithm is the FTA configured for products of linear factors.
- An alternate proof of Sylvester's Theorem is basically equivalent to computing the solution of constant-coefficient linear recurrence equations.
- If $d = 2s - 1$ and $r = s$, then the matrix is $s \times (s + 1)$ and has a non-trivial null-vector. The corresponding h (given in terms of the coefficients of p) has distinct factors unless its discriminant vanishes, giving the canonical form in odd degree.

9. Sylvester's canonical forms

- If $d = 2s$ and $r = s$, then the matrix is square, and in general, there exists λ so that $p(x,y) - \lambda x^{2s}$ has a matrix with a non-trivial null-vector as above, giving the canonical form in even degree.

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- In this even case, the determinant of the square matrix is the *catalecticant*. Sylvester apologized for introducing this term: "Meicatalecticizant would more completely express the meaning of that which, for the sake of brevity, I denominate the catalecticant." Sylvester was very interested in the technical aspects of poetry and a "catalectic" verse is one in which the last line is missing a foot.

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- To his credit, in the same paper, Sylvester introduced the term "unimodular" in its current meaning.

10. New steampunk canonical forms

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Theorem

Suppose $d \geq 1$, $\ell_j(x,y) = \beta_j x + \gamma_j y$, $1 \leq j \leq m$, *are fixed pairwise non-proportional linear forms and* e_k *is a proper divisor of* d, *and* $m + \sum_{k=1}^{r}(e_k + 1) = d + 1$. *Then a general binary form of degree* d *can be written as*

$$p(x,y) = \sum_{j=1}^{m} c_j \ell_j^d(x,y) + \sum_{k=1}^{r} f_k^{d/e_k}(x,y),$$

where $c_j \in \mathbb{C}$ *and* f_k *is a form of degree* e_k.

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This contains Sylvester's canonical forms, on taking $r = \lfloor d/2 \rfloor$ and $e_k \equiv 1$, so that $m = 0$ if d is odd and $m = 1$ if d is even.

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The novelty in these new canonical forms is the existence of forms of intermediate degree taken to intermediate powers.

10. New steampunk canonical forms

Proof.

The parameters are the m constants c_j and, for each k, the $e_k + 1$ coefficients of $f_k(x,y) = \sum_{u=0}^{e_k} \alpha_{ku} x^{e_k-u} y^u$.

The partials with respect to the c_j's are simply $\{\ell_1^d, \cdots, \ell_m^d\}$, and the partial with respect to α_{ku} is

$$(d/e_k) x^{e_k-u} y^u f_k^{d/e_k-1}.$$

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Now evaluate the Jacobian at a choice of parameters so that $f_k(x, y) = \tilde{\ell}_k^{e_k}$, where the linear forms $\tilde{\ell}_k$ are chosen so that the combined set $\{\ell_j, \tilde{\ell}_k\}$ is pairwise linearly independent. Then $f_k^{d/e_k - 1} = \tilde{\ell}_k^{d - e_k}$, and it is taken times a basis of $H_{e_k}(\mathbb{C}^2)$. By an earlier theorem, this set, taken all together, is a basis for $H_d(\mathbb{C}^2)$ and so this is a canonical form. □

10. New steampunk canonical forms

Clearly, the most interesting canonical forms of this kind occur when $m = 0$, so that $e_k \mid d$ and $\sum_{k=1}^{r}(e_k + 1) = d + 1$. Write $e_k m_k = d$, where $m_k > 1$ and observe that $\sum_{k=1}^{r}(e_k + 1) = d + 1 \implies r - 1 + \sum_{k=1}^{r} \frac{d}{m_k} = d \implies 1 = \sum_{k=1}^{r} \frac{1}{m_k} + \sum_{j=1}^{r-1} \frac{1}{d}$. It is well-known, and not hard to prove, that for any fixed r there are only finitely many such "Egyptian fraction" decompositions, hence for any r there are only finitely new canonical forms with r terms.

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For $r = 2$, there are three: $(d, e_1, e_2) = (3, 1, 1), (4, 2, 1), (6, 3, 2)$.

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For $r = 2$, there are three: $(d, e_1, e_2) = (3, 1, 1), (4, 2, 1), (6, 3, 2)$. For $r = 3$, there are 22, the most exotic of which is that a general binary form of degree 84 can be written as the square of a form of degree 42 plus the cube of a form of degree 28 plus the seventh power of a form of degree 12: $43 + 29 + 13 = 85$. I won't be looking for an algorithm very soon.

10. New steampunk canonical forms

If we further assume that all e_k's are equal, then we get a simple situation; $e \mid d$ and $e+1 \mid d+1$. Let $f(d)$ denote the number of $e < d$ with this property. We see that

$$d \equiv 0 \pmod{e}, \qquad d \equiv -1 \pmod{(e+1)}$$
$$\implies d \equiv e \pmod{(e^2+e)}$$

by the Chinese Remainder Theorem, hence $d = e + te(e+1)$ for $t \geq 1$, and standard generating function techniques give

$$\sum_{d=1}^{\infty} f(d)x^d = \sum_{e=1}^{\infty}\sum_{t=1}^{\infty} x^{e+te(e+1)} = \sum_{e=1}^{\infty} \frac{x^{e^2+2e}}{1-x^{e^2+e}}.$$

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It follows that if $F(N) = \sum_{d=1}^{N} f(d)$, then

$$F(N) = \sum_{e=1}^{\sqrt{N}} \left\lfloor \frac{N-e}{e^2+e} \right\rfloor \implies F(N) = N + \mathcal{O}(N^{1/2}).$$

10. New steampunk canonical forms

That is, the "average" number of these really nice canonical forms is about one per degree. Half of them are just the Sylvester canonical forms for odd degree $2k-1$, with $e=1$ and $r=k$. The first new one is $d=8$, $e=2$, $r=3$: a general binary octic is a sum of three quadratics to the fourth power. The first time $f(d)=2$ is $d=15$; the first time $f(d)=3$ is $d=99$.

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How large can $f(d)$ get? It's an amusing exercise to show that $f(p^m-1)=d(m)-1$, where $d(m)$ denotes the usual divisor function. This means that $(f(d))$ is unbounded and, taking $d=2^{2^u-1}$, $f(d)$ is of order at least $\log\log d$ infinitely often. Empirical evidence suggests that this is too modest and $\log d$ might be possible. For $d\le 10^7$, the maximum value is $f(7316000)=12$. The data also suggest that $(f(d))$ has an underlying distribution.

10. New steampunk canonical forms

We now look at specific instances of this new theorem.

- If $e_k \equiv 1$ (that is a mix of fixed and "open" linear forms to the d-th power), Sylvester's algorithm can be adapted to show uniqueness, again up to the order of the summands and roots of unity.

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- If $e_k \equiv 2$, an analogue to Sylvester's canonical forms occurs for general forms of even degree $d = 2k$: they are the sum of the k-th power of $\lfloor (d+1)/3 \rfloor$ quadratics plus a linear combination of any pre-specified $d - 3\lfloor (d+1)/3 \rfloor$ $2k$-th powers of linear forms. We don't have an algorithm for this. We want one. One problem is that it's easy to kill ℓ^d with a constant-coefficient linear differential operator; $q^{d/2}$, not so much.

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- If $d = 4$, $m = 0$, $e_1 = 2$ and $e_2 = 1$, a general binary quartic can be written as the sum of the square of a quadratic form and the fourth power of a linear form. (More later.)

10. New steampunk canonical forms

- If $d = 6$, $m = 0$, $e_1 = 3$ and $e_2 = 2$, then, as noted earlier, a general binary sextic form can be written as the sum of the square of a cubic form and the cube of a quadratic form. We don't have an algorithm for doing this and we (really)^2 want one! Mathematica computations suggest that sextic $p = f^2 + g^3$ for 40 different choices of $\{f^2, g^3\}$. Proofs would be welcomed.

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- If $d = 6$, $m = 0$, $e_1 = 3$ and $e_2 = 2$, then, as noted earlier, a general binary sextic form can be written as the sum of the square of a cubic form and the cube of a quadratic form. We don't have an algorithm for doing this and we $(\text{really})^2$ want one! Mathematica computations suggest that sextic $p = f^2 + g^3$ for 40 different choices of $\{f^2, g^3\}$. Proofs would be welcomed.
- Also, as noted earlier, a general binary form of degree $d = 2k$, $k \geq 1$, can be written as

$$(\lambda_0 x^2 + \lambda_1 xy + \lambda_2 y^2)^k + \sum_{j=1}^{k-1} (\alpha_j x + \beta_j y)^{2k}.$$

How many ways can this be done?

10. New steampunk canonical forms

If $d = 2k = 4$, it is not too hard to prove that $p = q^2 + (\alpha x + \beta y)^4$ for six different pairs of summands. It can also be shown that $\{\frac{\beta}{\alpha}\}$ is the image under a Möbius transformation of $\{0, \infty, \pm 1, \pm i\}$. (Since a general binary quartic is $x^4 + 6\lambda x^2 y^2 + y^4$ after a change of variables, it suffices to study this class of forms.)

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In general, for $p \in F_{2k}(\mathbb{C}^n)$, there exists quadratic q so that $p - q^k$ is a sum of $k - 1$ $2k$-th powers, and by Sylvester, this means that the $(k+2) \times k$ catalecticant matrix of $p - q^k$ has rank $k - 1$.

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Mathematica experiments show repeatedly that for $d = 2k = 6$ this happens in 22 ways, for $d = 2k = 8$ in 62 ways, for $d = 2k = 10$, in 147 ways, and for $d = 2k = 12$, the number is 308. With my clunky programming, this takes more than 24 hrs of CPU time: the number 308 has come up in four experiments and the kernel has run out of memory in three others.

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There actually is a pattern: $2\binom{k+3}{5} - \binom{k+2}{3}$.

12. Reichstein and canonically completing the cube

There is a wonderful non-trivial way to complete the cube, but almost nobody knows it. It appears in a paper by Boris Reichstein from 1987 which has no MathSciNet citations. It is a truly beautiful theorem, originally framed in the context of trilinear forms.

12. Reichstein and canonically completing the cube

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Here is more precise numerology about the number of cubes. By Alexander-Hirschowitz, for $n \neq 5$, a general cubic form in n variables can be written as a sum of $\lceil \frac{1}{n} N(n,3) \rceil = \left\lceil \frac{1}{n}\binom{n+2}{3} \right\rceil = \left\lceil \frac{(n+1)(n+2)}{6} \right\rceil$ cubes. (For $n = 5$, you need $\lceil \frac{6 \cdot 7}{6} \rceil + 1$ cubes.)

11. Reichstein and canonically completing the cube

Reichstein's Theorem writes a general cubic form as

$$\sum_{k=1}^{n} (\alpha_{k1} x_1 + \cdots + \alpha_{kn} x_n)^3 + q(x_3, \ldots, x_n).$$

After iterating the construction, one finds that a general cubic is a sum of $\sum_{0 \le k \le n/2} (n - 2k) = \lfloor \frac{(n+1)^2}{4} \rfloor$ cubes, which is, on average, about 50% larger than what is necessary.

11. Reichstein and canonically completing the cube

Reichstein's Theorem writes a general cubic form as

$$\sum_{k=1}^{n}(\alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n)^3 + q(x_3, \ldots, x_n).$$

After iterating the construction, one finds that a general cubic is a sum of $\sum_{0\le k\le n/2}(n-2k) = \lfloor \frac{(n+1)^2}{4} \rfloor$ cubes, which is, on average, about 50% larger than what is necessary.

But $N(n,3) - N(n-2,3) = \frac{n^3+3n^2+2n}{6} - \frac{n^3-3n^2+2n}{6} = n^2$, so that the total number of coefficients is

$$\sum_{0\le k\le n/2}(n-2k)^2 = N(n,3),$$

showing a potential canonical form. For $n=5$, $N(5,3)=35$ and $35 = 5+5+5+5+5+5+5$ but a quinary cubic isn't a sum of 7 cubes of linear forms; via Reichstein, $35 = 5+5+5+5+5+3+3+3+1$ and it is a sum of 9 cubes canonically.

11. Reichstein and canonically completing the cube

Lasker-Wakeford, specializing at $x_1, x_2, x_1 + kx_2 + x_k$ (for $k \geq 3$) for linear forms in $(x_1, \ldots, x_n)$, etc., can be used to give a non-constructive proof, but Reichstein's constructive proof is better.

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We need the well-known fact that a general pair of quadratic forms can be simultaneously diagonalized. That is, if general $f, g \in H_2(\mathbb{C}^n)$ are given, then there exist n linearly independent forms $L_i(x) = \sum_{j=1}^n \alpha_{ij} x_j$ and $c_i \in \mathbb{C}$ so that

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This can be done constructively. If $rank(f) = n$ and the determinant of the symmetric matrix associated with the pencil $f - \lambda g$ has n distinct roots $\{c_i\}$, then each $f - c_i g$ is singular. Routine methods can then be used to find the L_i's.

11. Reichstein and canonically completing the cube

We now prove Reichstein's Theorem. Suppose $p \in H_3(\mathbb{C}^n)$. We can generally simultaneously diagonalize $\frac{\partial p}{\partial x_1}$ and $\frac{\partial p}{\partial x_2}$: there exist linearly independent $L_i(x) = \sum_{j=1}^n \alpha_{ij} x_j$ and $c_i \in \mathbb{C}$ so that

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Since mixed partials are equal, we obtain the equation

$$\sum_{i=1}^n 2\alpha_{i2} L_i = \sum_{i=1}^n 2 c_i \alpha_{i1} L_i,$$

and since the L_i's are linearly independent, $\alpha_{i2} = c_i \alpha_{i1}$. (This is important!)

11. Reichstein and canonically completing the cube

As before, it is generally true that $\alpha_{i1} \neq 0$ and we can let

$$q(x_1, \ldots, x_n) = p(x_1, \ldots, x_n) - \sum_{i=1}^{n} \frac{1}{3\alpha_{i1}} L_i^3$$

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$$\implies q = q(x_3,\dots,x_n).$$

By iterating, we obtain Reichstein's form for cubics:

$$p(x_1,\dots,x_n) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \sum_{j=1}^{n-2i} \ell_{ij}^3(x_{1+2i},\dots,x_n).$$

12. Slinky

Recall the definition of Slinky:

$$p(x_1, \ldots, x_n) = \sum_{1 \le i \le j \le n} (\alpha_{\{i,j\},i} x_i + \cdots + \alpha_{\{i,j\},j} x_j)^3.$$

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Slinky is potentially canonical, because the coefficients $\alpha_{\{i,j\},k}$ are parameterized by $1 \le i \le k \le j \le n$, and so there are $\binom{n+2}{3}$ of them. You can probably guess by now how it's going to be proved.

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$$\frac{\partial p}{\partial x_n} = \sum_{j=1}^{n} (\alpha_{jj} x_j + \cdots + \alpha_{jn} x_n)^2.$$

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Let

$$q(x_1, \dots, x_n) = p(x_1, \dots, x_n) - \sum_{j=1}^{n} \frac{1}{3\alpha_{jn}} (\alpha_{jj} x_j + \cdots + \alpha_{jn} x_n)^3.$$

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Then

$$\frac{\partial q}{\partial x_n} = \frac{\partial p}{\partial x_n} - \frac{\partial p}{\partial x_n} = 0 \implies q = q(x_1, \dots, x_{n-1}).$$

and repeat. We assume $\alpha_{jn} \neq 0$, etc., which is generally true. In this way, for each pair (i, j) with $1 \le i \le j \le n$, we get exactly one summand using only the x_k's with $i \le k \le j$.

We could also give an explicit construction of *every* cubic form in n variables as a sum of $\binom{n+1}{2}$ cubes of linear forms, but this talk is too long as it is.

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This last construction worked because in the upper diagonal sum of squares for quadratic forms, there is a variable, x_n, which appears in every summand. This is not the case for the cubic version, so there is no obvious way to bump it up to quartics.

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The Reichstein form, on the other hand, **can** be generalized to quartics, in the same way, by integrating on the coefficient of x_n. One gets a general $p \in H_4(\mathbb{C}^n)$ as a sum of $\sum_{j=0}^{n} \frac{(n+1-j)^2}{4} \approx \frac{1}{12} n^3$ fourth powers, which is about twice the minimal number. But this quartic version has no universally-used variable, so it can't be bumped up to the fifth power.

13. Another number-theoretic digression

Imagine a general canonical form for quartics of "Reichstein-type"

$$p(x_1, \dots, x_n) = \sum_{k=1}^{r} (\alpha_{k1} x_1 + \cdots + \alpha_{kn} x_n)^4 + q(x_1, \dots, x_m).$$

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It turns out that if $n = 12$, there does **not** exist $m < 12$ so that

$$12 \;\Big|\; \binom{15}{4} - \binom{m+3}{4},$$

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More generally, these are ruled out when n belongs to the set

$$A_d = \left\{ n : 0 \le m < n \implies n \not| \binom{n+d-1}{d} - \binom{m+d-1}{d} \right\}.$$

We have a few partial results.

- If $3 \not| k$, then $n = 2^{2k} \cdot 3 \in A_4$.

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- If $3 \nmid k$, then $n = 2^{2k} \cdot 3 \in A_4$.
- If $p \equiv 1 \pmod{144}$ is prime, then $12p \in A_4$.

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- If p is prime, then $p \mid \binom{n+p-1}{p} - \binom{n}{p}$, hence A_p is empty for prime p (such as $p = 2, 3$) and there is no obstacle in prime degree for Reichstein-type canonical forms.

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- The smallest elements of $A_6, A_8, A_{10}, A_{12}, A_{14}$ and A_{15} are 10, 1792, 6, 242, 338 and 273 respectively. If A_9 or A_{16} are non-empty, then their smallest elements are at least 10^5. (Fortunately, steampunk allows Mathematica.)

14. Other kinds of canonical forms

It seems obvious to 21st century mathematicians, if not 19th century mathematicians, that one should look at polynomial maps $F : S \mapsto H_d(\mathbb{C}^n)$, where S is an N-dimensional subspace of $\mathbb{C}^M$ for some $M > N$.

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Theorem

For fixed $\{r_k\}$, a general binary quadratic form can be written as

$$(t_1 x + t_2 y)^2 + (t_3 x + t_4 y)^2, \quad \textit{where} \quad \sum_{k=1}^{4} r_k t_k = 0$$

unless $r_3 = \pm i r_1$ and $r_2 = \pm i r_4$; that is, unless there exists (x_0, y_0) so that $t_1 x_0 + t_2 y_0 = i(t_3 x_0 + t_4 y_0)$ on the hyperplane.

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This is proved by parameterizing the plane and looking at the Jacobian. The choice $(r_1, r_2, r_3, r_4) = (1, 0, i, 0)$ gives the "silly" example from the introduction.

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One final canonical form: We have proved the following result for $d = 2k = 2, 4, 6, 8$ and believe it is true in general. If true, it would provide another complement to Sylvester's canonical form for even degree:

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A general binary form of degree $d = 2k$ can be written as

$$p(x, y) = \sum_{j=1}^{k+1} (\alpha_j x + \beta_j y)^{2k}, \quad \text{where} \quad \sum_{j=1}^{k+1} (\alpha_j + \beta_j) = 0.$$

In the cases given, the Jacobian is non-zero when evaluated at a point whose coordinates are mostly small positive consecutive integers.

15. To the audience

Thanks for your patience and for coming to the talk!

16. Oh, good. I have some extra time