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Lattice Polytopes with Distinct Pair-Sums

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Abstract. Let P be a lattice polytope in R^n, and let $P \cap Z^n = \{v_1, \ldots, v_N\}$. If the $N + \binom{N}{2}$ points $2v_1, \ldots, 2v_N$; $v_1 + v_2, \ldots, v_{N-1} + v_N$ are distinct, we say that P is a "distinct pair-sum" or "dps" polytope. We show that if P is a dps polytope in R^n, then $N \le 2^n$, and, for every n, we construct dps polytopes in R^n which contain 2^n lattice points. We also discuss the relation between dps polytopes and the study of sums of squares of real polynomials.

Let $\mathcal{P}$ be a lattice polytope in $\mathbb{R}^n$, the convex hull of a finite set in $\mathbb{Z}^n$, and let

$$\mathcal{L}(\mathcal{P}) := \mathcal{P} \cap \mathbb{Z}^n = \{v_1, \ldots, v_N\},$$

where $N = N(\mathcal{P}) := |\mathcal{L}(\mathcal{P})|$. Suppose the $N + \binom{N}{2}$ points in $\mathcal{L}(\mathcal{P}) + \mathcal{L}(\mathcal{P})$,

$$2v_1, \ldots, 2v_N;\ v_1 + v_2,\ v_1 + v_3, \ldots, v_{N-1} + v_N,$$

are distinct. In this case we say that $\mathcal{P}$ is a *distinct pair-sum* or *dps* polytope. Our interest in dps polytopes comes from the study of the representation of polynomials as a sum of squares of polynomials.

The following lemma offers two other geometrical characterizations of dps polytopes.

Lemma 1. *Let $\mathcal{P}$ be a lattice polytope. Then the following are equivalent*:

(1) $\mathcal{L}(\mathcal{P})$ *is a dps polytope.*

(2) *$\mathcal{L}(\mathcal{P})$ does not contain the vertices of a (nondegenerate) parallelogram, and does not contain three collinear points.*

(3) *Suppose $v \neq v'$ and $w \neq w'$ are in $\mathcal{L}(\mathcal{P})$. Then $v' - v$ and $w' - w$ are parallel only if $\{v, v'\} = \{w, w'\}$.*

Proof. (1) $\Rightarrow$ (2) Suppose $v_1, v_2, v_3, v_4 \in \mathcal{L}(\mathcal{P})$ are the vertices of a parallelogram. Then $v_1 - v_2 = v_3 - v_4$ implies $v_1 + v_4 = v_2 + v_3$, so that $\mathcal{P}$ is not dps. Now suppose $v_1, v_2, v_3 \in \mathcal{L}(\mathcal{P})$, and v_2 is interior to the line segment $\overline{v_1 v_3}$. If v_2 is the midpoint of the segment, then $v_2 + v_2 = v_1 + v_3$, so $\mathcal{P}$ is not dps. Otherwise, we may assume that v_2 is closer to v_1 than to v_3. Then $v_4 = v_2 + (v_2 - v_1)$ will also be a lattice point on the line segment $\overline{v_1 v_3}$, and v_2 is the midpoint of $\overline{v_1 v_4}$; again, $\mathcal{P}$ is not dps.

(2) $\Rightarrow$ (3) For $u \in \mathbb{Z}^n$, let $g(u) = \gcd(u_1, \dots, u_n)$. Suppose $g(u' - u) = d > 1$. Then $u' - u = du''$ for $u'' \in \mathbb{Z}^n$, and the line segment $\overline{uu'}$ contains the lattice points $u, u + u'', \dots, u + du'' = u'$. Thus, if (2) holds and $u, u' \in \mathcal{L}(\mathcal{P})$, $u \neq u'$, we have $g(u' - u) = 1$. Suppose $w' - w = \alpha \cdot (v' - v)$. Then $\alpha = p/q$ for nonzero integers p, q, and $q(w' - w) = p(v' - v)$. Hence $|q| = g(q(w' - w)) = g(p(v' - v)) = |p|$, so $\alpha = \pm 1$. Now the parallelogram condition in (2) implies that $\{v, v'\} = \{w, w'\}$.

(3) $\Rightarrow$ (1) If (3) holds for $\mathcal{P}$, and $v_i, v_j, v_k, v_\ell \in \mathcal{L}(\mathcal{P})$ with $i \notin \{k, \ell\}$, then $v_i - v_k \neq v_\ell - v_j$, and so $v_i + v_j \neq v_k + v_\ell$. This proves (1). □

Our main results are these: if $\mathcal{P}$ in $\mathbb{R}^n$ is a dps polytope, then $N(\mathcal{P}) \leq 2^n$, and, for every n, we construct dps polytopes in $\mathbb{R}^n$ for which $N(\mathcal{P}) = 2^n$.

Example 1. Let $\mathcal{P} \subset \mathbb{R}^2$ be the triangle with vertices $\{(0, 1), (1, 2), (2, 0)\}$. Then $\mathcal{P}$ is a dps polytope, because

$$\mathcal{L}(\mathcal{P}) = \{(0, 1), (1, 2), (2, 0), (1, 1)\}$$

and

$$\mathcal{L}(\mathcal{P}) + \mathcal{L}(\mathcal{P}) = \{(0, 2), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (4, 0)\}.$$

We can view $\mathcal{P}$ as the projection onto the first two coordinates of the triangle with vertices $\{(0, 1, 2), (1, 2, 0), (2, 0, 1)\}$, which lies in the hyperplane $x_1 + x_2 + x_3 = 3$. (In this example, we could have just as well taken the triangle with vertices $\{(0, 0), (1, 2), (2, 1)\}$; again, $\mathcal{L}(\mathcal{P})$ will consist of the vertices of $\mathcal{P}$ and $(1, 1)$.)

Example 2. Let

$$\mathcal{A} = \{(4, 1, 0, 0), (0, 4, 1, 0), (0, 0, 4, 1), (1, 0, 0, 4)\},$$
$$\mathcal{B} = \{(2, 1, 1, 1), (1, 2, 1, 1), (1, 1, 2, 1), (1, 1, 1, 2)\};$$

and let $\mathcal{P} = \operatorname{conv}(\mathcal{A} \cup \mathcal{B}) \subset \mathbb{R}^4$ be the convex hull of $\mathcal{A} \cup \mathcal{B}$. By construction, $\mathcal{P}$ is cyclically symmetric with respect to its coordinates. It is not hard to show that $\mathcal{L}(\mathcal{P}) = \mathcal{A} \cup \mathcal{B}$. Suppose $w = (w^{(1)}, w^{(2)}, w^{(3)}, w^{(4)}) \in \mathcal{L}(\mathcal{P})$. Since w is a convex combination of $\mathcal{A} \cup \mathcal{B}$, we have $w^{(i)} \geq 0$ and $\sum_i w^{(i)} = 5$. If $w^{(i)} \geq 1$ for all i, then w must be a permutation of $(2, 1, 1, 1)$ and so lies in $\mathcal{B}$. Otherwise, $w^{(i)} = 0$ for some i, and by cycling the coordinates, we may assume that $w^{(1)} = 0$. However, then w must be a convex combination of $(0, 4, 1, 0)$ and $(0, 0, 4, 1)$ and so $w \in \mathcal{A}$. A routine check, which we omit, shows that the $8 + \binom{8}{2} = 36$ sums in $\mathcal{L}(\mathcal{P}) + \mathcal{L}(\mathcal{P})$ are distinct. By projecting $\mathcal{P}$ onto its first three coordinates, we obtain a dps polytope in $\mathbb{R}^3$ with $N(\mathcal{P}) = 8$.

Theorem 2. *Suppose $\mathcal{P}$ is a dps polytope in $\mathbb{R}^n$. Then $N(\mathcal{P}) \leq 2^n$.*

Proof. If $N(\mathcal{P}) > 2^n$, then, by the Pigeonhole Principle, there exist $v_i \neq v_j$ so that v_i and v_j are componentwise congruent modulo 2. This means that $v_k = \frac{1}{2}(v_i + v_j) = v_i + \frac{1}{2}(v_j - v_i)$ is also a lattice point, and it follows from Lemma 1 that $\mathcal{P}$ is not a dps polytope. □

This argument is essentially the same one used to solve Putnam Problem 1971-A1 (see [1]): "Let there be given nine lattice points (points with integral coordinates) in three dimensional Euclidean space. Show that there is a lattice point on the interior of one of the line segments joining two of these points." The proof of Theorem 2 also applies to the less restrictive class of convex polytopes which do not contain three lattice points on a line. One such polytope is the n-cube $\mathcal{C}_n = \{0, 1\}^n$, which has many lattice parallelograms.

We say that a dps polytope $\mathcal{P} \subset \mathbb{R}^n$ for which $N(\mathcal{P}) = 2^n$ is *maximal*. The proof of Theorem 2 implies that no two points in a dps polytope are componentwise congruent modulo 2; hence a maximal dps polytope contains one representative from every congruence class modulo 2 (and at most one representative from every congruence class modulo m, $m \geq 3$).

Suppose M is an $n \times n$ unimodular matrix with integer entries. Then M defines a linear mapping on $\mathbb{R}^n$ (viewed as column vectors) by matrix multiplication. Since linear mappings preserve inclusions and both M and M^{-1} have integer entries, it is easy to see that $\mathcal{L}(M(\mathcal{P})) = M(\mathcal{L}(\mathcal{P}))$ for any lattice polytope $\mathcal{P}$, and since linear mappings preserve sums, it is then clear that $\mathcal{P}$ is dps if and only if $M(\mathcal{P})$ is dps.

Theorem 3. *There exist maximal dps polytopes in $\mathbb{R}^n$ for every n.*

Proof. For $n = 1$, let $\mathcal{P} = [0, 1]$; for $n = 2, 3$, consider Examples 1 and 2. Suppose now that $\mathcal{P}$ is a maximal dps polytope in $\mathbb{R}^n$, $n \geq 3$. Write $\mathcal{L} = \mathcal{L}(\mathcal{P})$ and define the (finite) set of differences

$$\mathcal{D} = (\mathcal{L} - \mathcal{L})^* := \{v - v' \colon v, v' \in \mathcal{L}, v \neq v'\}.$$

Let M be a unimodular integer matrix such that if $u \in \mathcal{D}$, then $M(u) \notin \mathcal{D}$. (We construct such an M below.)

We define the polytope $\mathcal{P}'$ in $\mathbb{R}^{n+1}$ as follows. Let

$$\mathcal{A} = \{(v, 0) \in \mathbb{R}^{n+1} \colon v \in \mathcal{L}(\mathcal{P})\}, \qquad \mathcal{B} = \{(M(v), 1) \in \mathbb{R}^{n+1} \colon v \in \mathcal{L}(\mathcal{P})\},$$

and let $\mathcal{P}' = \mathrm{conv}(\mathcal{A} \cup \mathcal{B})$. If $w = (w^{(1)}, \ldots, w^{(n+1)}) \in \mathcal{L}(\mathcal{P}')$, then $0 \leq w^{(n+1)} \leq 1$, hence $w^{(n+1)}$ equals 0 or 1. Thus, w lies either on the face determined by $\mathcal{A}$, in which case $w = (v, 0)$, or on the face determined by $\mathcal{B}$, in which case $w = (M(v), 1)$. It follows that $\mathcal{L}(\mathcal{P}') = \mathcal{A} \cup \mathcal{B}$, so $N(\mathcal{P}') = 2^{n+1}$.

Now consider $\mathcal{L}(\mathcal{P}') + \mathcal{L}(\mathcal{P}')$; this consists of three disjoint sets of points:

$$\{(v_i, 0) + (v_j, 0)\}, \quad \{(v_i, 0) + (M(v_j), 1)\}, \quad \{(M(v_i), 1) + (M(v_j), 1)\},$$

where $v_i, v_j \in \mathcal{L}(\mathcal{P})$. Since both $\mathcal{P}$ and $M(\mathcal{P})$ are dps, the sums in the first and the third set are distinct. For the second set, we suppose that

$$(v_i, 0) + (M(v_j), 1) = (v_k, 0) + (M(v_\ell), 1), \tag{1}$$

or, equivalently,

$$v_i - v_k = M(v_\ell) - M(v_j) = M(v_\ell - v_j).$$

If $j = \ell$, then $v_i - v_k = 0$, so $i = k$, which is the only possible way for (1) to hold in a dps polytope. Otherwise, $j \neq \ell$, so $M(v_\ell - v_j) = v_i - v_k \in \mathcal{D}$, a contradiction to the choice of M. Thus, $\mathcal{P}'$ is a maximal dps polytope in $\mathbb{R}^{n+1}$.

We now construct a matrix M with the desired properties. First, let

$$R = \max\{|u_j^{(k)}|\colon\ u_j \in \mathcal{D}, 1 \le k \le n\},$$

and let M be the $n \times n$ matrix given below:

$$M = \begin{pmatrix} 1+(R+1)^2 & R+1 & 0 & 0 & \cdots & 0 & 0 \\ R+1 & 1 & R+1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & R+1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & R+1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

(In words, the only nonzero entries in M are the diagonal, those just above the diagonal, and the first entry in the second row.) It is easy to see that M is unimodular.

We show now that for every $u \in \mathcal{D}$, at least one entry of $w = M(u)$ has absolute value greater than R. This implies that $M(u) \notin \mathcal{D}$, and will complete the proof. Write $u = (u^{(1)}, \ldots, u^{(n)})$ and suppose that k is the smallest index such that $u^{(k)} \neq 0$. (Such an index exists because $0 \notin \mathcal{D}$.)

If $k = 1$, then $w^{(1)} = (1 + (R+1)^2)u^{(1)} + (R+1)u^{(2)}$, and hence

$$|w^{(1)}| \ge |(1+(R+1)^2)u^{(1)}| - (R+1)|u^{(2)}| \ge 1 + (R+1)^2 - R(R+1) = R+2.$$

If $k \ge 2$, then $u^{(1)} = \cdots = u^{(k-1)} = 0$, so $w^{(k-1)} = (R+1)u^{(k)}$ and $|w^{(k-1)}| \ge R+1$. Finally, we remark that the same proof applies in the case $n = 2$, if we take as our matrix the 2×2 submatrix at the upper left of M. □

Example 3. We illustrate the last construction by applying it to the polytope in Example 1, for which

$$\mathcal{D} = \{\pm(0,1), \pm(1,-2), \pm(1,-1), \pm(1,0), \pm(1,1), \pm(2,-1)\},$$

so $R = 2$ and

$$M = \begin{pmatrix} 10 & 3 \\ 3 & 1 \end{pmatrix}.$$

Thus, $\mathrm{conv}(\mathcal{A} \cup \mathcal{B})$ is a maximal dps polytope in $\mathbb{R}^3$, where

$$\mathcal{A} = \{(0,1,0), (1,1,0), (1,2,0), (2,0,0)\},$$
$$\mathcal{B} = \{(3,1,1), (13,4,1), (16,5,1), (20,6,1)\}.$$

We could now apply the shear $(x_1, x_2, x_3) \mapsto (x_1 - 3x_2 - 5x_3 + 5, x_2 - x_3, x_3)$, which maps $\mathcal{A}$ and $\mathcal{B}$ to

$$\mathcal{A}' := \{(2,1,0), (3,1,0), (0,2,0), (7,0,0)\}$$

and

$$\mathcal{B}' := \{(0,0,1), (1,3,1), (1,4,1), (2,5,1)\},$$

respectively, in order to reduce the magnitude of the coordinates in the example.

Since any translate of a dps polytope is also dps, we may always assume, as we have done in the examples, that $\mathcal{P}$ lies in the nonnegative orthant of $\mathbb{R}^n$. In this case, we define $s(\mathcal{P})$, the *size* of $\mathcal{P}$:

$$s(\mathcal{P}) = \max\{v_j^{(1)} + \cdots + v_j^{(n)}\colon\ v_j \in \mathcal{L}(\mathcal{P})\}.$$

If $s = s(\mathcal{P})$, then $\mathcal{P}$ can be viewed as a projection onto the first n coordinates of a polytope in $\mathbb{R}^{n+1}$ which lies in the simplex

$$\Delta_{n+1}(s) := \left\{ u = (u^{(1)}, \ldots, u^{(n+1)})\colon\ u^{(i)} \geq 0,\ \sum_{i=1}^{n+1} u^{(i)} = s \right\}.$$

Let s_n denote the minimum size of any maximal dps polytope in $\mathbb{R}^n$. Examples 1 and 2 show that $s_2 \leq 3$ and $s_3 \leq 5$. It is not difficult to show that these estimates are sharp. The first case can be done by hand: if $\mathcal{P}$ is a maximal dps polytope with size 2 in $\mathbb{R}^2$, then $\mathcal{L}(\mathcal{P})$ must consist of four points chosen from

$$\{(0,0), (0,1), (0,2), (1,0), (1,1), (2,0)\}.$$

Since each congruence class is represented in $\mathcal{L}(\mathcal{P})$, it must contain $(0,1)$, $(1,0)$, and $(1,1)$. These three points form a parallelogram with each of the points $(0,0)$, $(0,2)$, and $(2,0)$. Hence no fourth point can exist in $\mathcal{L}(\mathcal{P})$ while preserving the dps property. The second case is similar, but much more complicated. Computer-aided calculations can be used to conclude that no dps polytope in $\mathbb{R}^3$ has size 4 or less. (We thank Dr. Bruce Carpenter for doing the Mathematica coding.)

It can also be shown, using the style of argument of Section 3 of [6], that every maximal dps polytope in $\mathbb{R}^2$ is the image of the triangle in Example 1 under an affine unimodular linear mapping, and consists of a triangle with area $\frac{3}{2}$, and a single lattice point inside, which will always be the centroid of the triangle. The tetrahedron determined by $\mathcal{B}$ in Example 2 lies within the tetrahedron determined by $\mathcal{A}$, whereas in Example 3, each point in $\mathcal{L}$ is on the boundary of the polytope. Thus there are at least two distinct combinatorial types of maximal dps polytopes in $\mathbb{R}^3$.

We make no serious conjecture about the growth of s_n. On the one hand, any maximal dps polytope must contain a lattice point with odd coordinates, so $s_n \geq n$. In the other direction, it is not difficult to use the proof of Theorem 3 to obtain a doubly exponential bound for s_n. Since this bound is likely to be very crude, we do not present it explicitly. Another open question is to determine the minimum volume of a maximal dps polytope in $\mathbb{R}^n$ for $n \geq 3$. We also do not know the answer to the following question: is every dps polytope a subset of a maximal dps polytope?

We now discuss our original interest in this subject. Given $u \in \mathbb{Z}_+^n$, define the monomial $x^u \in \mathbb{R}[x_1, \ldots, x_n]$ by

$$x^u = x_1^{u^{(1)}} \cdots x_n^{u^{(n)}}.$$

Suppose $\mathcal{U} \subseteq \mathbb{Z}_+^n$ and consider the polynomial

$$p(x_1, \ldots, x_n) = \sum_{u \in \mathcal{U}} b_u x^u.$$

In [4], the present authors developed an algorithm for determining whether p can be written as a sum of squares of polynomials. A necessary condition is that p is psd; that is, $p(x_1, \ldots, x_n) \geq 0$ for all $x \in \mathbb{R}^n$. Suppose p is psd and let

$$\mathcal{C}(p) = \operatorname{conv}\{u\colon\ b_u \neq 0\}.$$

Then $\mathcal{C}(p)$ is a lattice polytope; in fact it can be shown that the vertices of $\mathcal{C}(p)$ lie in $(2\mathbb{Z})^n$, so that $\mathcal{P} := \frac{1}{2}\mathcal{C}(p)$ is a lattice polytope. Let

$$\mathcal{L}(\mathcal{P}) = \{v_1, \ldots, v_N\},$$

and for $u \in \mathcal{C}(p)$, let $D(u) = \{(i, j)\colon\ v_i + v_j = u\}$. It is proved in Theorem 2.4 of [4] that p can be written as a sum of at most r squares of polynomials if and only if there is a real $N \times N$ symmetric psd matrix $A = [a_{ij}]$ of rank at most r, so that

$$\sum_{(i,j) \in D(u)} a_{ij} = b_u \qquad \text{for all} \quad u \in \mathcal{C}(p).$$

If $\mathcal{P}$ is a dps polytope in $\mathbb{R}^n$, then either $|D(u)| \leq 1$ or $D(u) = \{(i, j), (j, i)\}$. In either case, a_{ij} is completely determined by b_u. In particular, if

$$h_{\mathcal{P}}(x_1, \ldots, x_n) := \sum_{i=1}^{N} (x^{v_i})^2, \tag{2}$$

then A must equal I_N, the $N \times N$ identity matrix, so that p is a sum of N squares, and no fewer.

Finally, we note that the homogenization of polynomials with n variables into forms with $n + 1$ variables is precisely analogous to the embedding of polytopes in $\mathbb{R}_+^n$ into the hyperplane $\Delta_{n+1}(s)$.

Example 4. (See Example 3.9 of [4].) We return to Example 1, in its homogeneous version. Let $A = [a_{ij}]$ be a real symmetric 4×4 matrix and let

$$f(t_1, t_2, t_3, t_4) = \sum_{i=1}^{4} \sum_{j=1}^{4} a_{ij} t_i t_j$$

be its associated quadratic form. We use the substitution suggested by $\mathcal{L}(\mathcal{P})$ and define the ternary sextic form

$$p(x_1, x_2, x_3) = f(x_2 x_3^2, x_1 x_2^2, x_1^2 x_3, x_1 x_2 x_3).$$

Then p is a sum of squares of polynomials (cubic forms) if and only if f is a psd quadratic form; that is,

$$f(t_1, t_2, t_3, t_4) \geq 0 \qquad \text{for all} \quad (t_1, t_2, t_3, t_4) \in \mathbb{R}^4.$$

Since $t_4^3 = t_1 t_2 t_3$, the condition for p to be a psd form is weaker:

$$f(t_1, t_2, t_3, (t_1 t_2 t_3)^{1/3}) \geq 0 \qquad \text{for all} \quad (t_1, t_2, t_3) \in \mathbb{R}^3.$$

If $f(t_1, t_2, t_3, t_4) = t_1^2 + t_2^2 + t_3^2 - 3t_4^2$, then f is not psd, but $f(t_1, t_2, t_3, (t_1 t_2 t_3)^{1/3}) \geq 0$ by the arithmetic-geometric inequality. It follows that

$$p(x_1, x_2, x_3) = x_2^2 x_3^4 + x_1^2 x_2^4 + x_1^4 x_3^2 - 3x_1^2 x_2^2 x_3^2$$

is a form which is psd, but not a sum of squares of polynomials. This particular example was discussed in [3]. For a history and bibliography of this subject and its relation to Hilbert's 17th Problem, see [7].

More generally, the *Pythagoras number* of a ring A, $P(A)$, is the smallest number $n \leq \infty$ such that any sum of squares in A can be expressed as a sum of at most n squares in A. Pfister [5] proved in 1967 that $P(\mathbb{R}(x_1, \ldots, x_n)) \leq 2^n$. It is easy to see that $P(\mathbb{R}[x_1]) = 2$. Since maximal dps polytopes exist in $\mathbb{R}^n$ for every n, a consideration of $h_{\mathcal{P}}$ (see (2)) shows that $P(\mathbb{R}[x_1, \ldots, x_n]) \geq 2^n$. This is not the strongest result possible: on p. 60 of [2], using other methods, Dai and the present authors have shown that $P(\mathbb{R}[x_1, \ldots, x_n]) = \infty$ for $n \geq 2$.

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