Math 347 Bonus Notes 13 (For the Brave) Fri., Nov. 16, 2001

(None of this will be on a test, but it might help with the last problem in the homework.) Let's start with a theorem I've mentioned a number of times, but first a definition. a *rearrangement* π of $\mathbf{N}$ is a one-to-one onto map of $\mathbf{N}$ to itself; that is, a sequence of integers $\{\pi(1), \pi(2), \ldots\}$ with the property that for every $i \in \mathbf{N}$, there is exactly one $j \in \mathbf{N}$ so that $\pi(j) = i$. This j is called $\pi^{-1}(i)$. Examples of rearrangements of $\mathbf{N}$ are the sequences $\{2, 1, 4, 3, 6, 5, 8, 7, \ldots\}$ and $\{1, 3, 2, 5, 7, 4, 9, 11, 6, \ldots\}$.

Theorem:. Suppose $\sum a_n$ is an absolutely convergent series and $\sum a_n = L$. Then for every rearrangement π, $\sum a_{\pi(n)}$ is also convergent, with the *same* sum: L.

Remark: The property of absolute convergence is necessary. We have seen that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - + \ldots$ can be rearranged to sum to infinitely many different limits. (Recall that if you take r positive and s negative terms alternately, you wind up with $\frac{1}{2}\log(4r/s)$.)

Proof: Let π be given and let $\epsilon > 0$ be given. Since $\sum |a_n|$ converges, there exists an integer K so that

$$\sum_{n=K+1}^{\infty} |a_n| < \epsilon.$$

Let $N = \max\{\pi^{-1}(1), \ldots, \pi^{-1}(K)\}$. This means that the set $\{\pi(1), ..., \pi(N)\}$ contains, possibly among other elements, $\{1, ..., k\}$. If $k > N$, we must have $\pi(k) > K$. We now want to show that $\sum a_{\pi(n)}$ is Cauchy. Suppose $m > n > N$. Then for $m \geq k \geq n+1$, we have $\pi(k) > K$ and the $\pi(k)$'s are distinct, so

$$\left|\sum_{k=m+1}^{n} a_{\pi(k)}\right| \leq \sum_{k=m+1}^{n} |a_{\pi(k)}| < \sum_{j=K+1}^{\infty} |a_j| < \epsilon.$$

This proves that $\sum a_{\pi(n)}$ is convergent, to say L'. But how do we know that $L = L'$? If $m \geq K$, then

$$\left|L - \sum_{k=1}^{m} a_k\right| = \left|\sum_{k=m+1}^{\infty} a_k\right| \leq \sum_{k=m+1}^{\infty} |a_k| < \sum_{n=K+1}^{\infty} |a_n| < \epsilon.$$

If $n > N$, then

$$\sum_{k=1}^{n} a_{\pi(k)} - \sum_{k=1}^{K} a_k$$

is a sum of $n - K$ terms of the form a_j, where $j > K$, because each of the terms in the second sum appears also in the first sum. Hence, if we take absolute values we find that

$$\left|\sum_{k=1}^{n} a_{\pi(k)} - \sum_{k=1}^{K} a_k\right| \leq \sum_{n=K+1}^{\infty} |a_n| < \epsilon.$$

It follows immediately that

$$\left|\sum_{k=1}^{n} a_{\pi(k)} - L\right| < 2\epsilon.$$

and since this is true for arbitary ϵ, we must have $L = L'$. •

This theorem has a useful corollary. We say that X and Y *partition* $\mathbf{N}$ if $X \cup Y = \mathbf{N}$ and $X \cap Y = \emptyset$; that is, for every $n \in \mathbf{N}$ either $n \in X$ or $n \in Y$, but not both.

Corollary: Suppose $\sum a_n$ is absolutely convergent. Then for every partition of $\mathbf{N}$,

$$\sum_{n \in \mathbf{N}} a_n = \sum_{n \in \mathbf{X}} a_n + \sum_{n \in \mathbf{Y}} a_n.$$

Proof: First observe that $\sum_{n\in X} |a_n| \le \sum_{n\in\mathbf{N}} |a_n|$, hence $\sum_{n\in X} a_n$ is absolutely convergent, and so is convergent. The same holds for the other set. Let $L = \sum_{n\in\mathbf{N}} a_n$, $L_1 = \sum_{n\in\mathbf{X}} a_n$ and $L_2 = \sum_{n\in\mathbf{Y}} a_n$. For clarity, let $X = \{x_1 < x_2 < x_3 < \ldots\}$ and $Y = \{y_1 < y_2 < y_3 < \ldots\}$ and consider the rearrangement $\pi = \{x_1, y_1, x_2, y_2, \ldots\}$. Consider the partial sums

$$s_N = \sum_{n=1}^{N} a_{x_n}, \qquad t_N = \sum_{n=1}^{N} a_{y_n}.$$

Then $s_N \to L_1$ and $t_N \to L_2$, but $s_N + t_N$ is the partial sum

$$\sum_{n=1}^{2N} a_{\pi(n)},$$

which converges to L by the Theorem, hence $L = L_1 + L_2$. •

It is easy to generalize the definition to a partition of $\mathbf{N}$ into $r > 2$ sets; the analogous theorem for the sums applies by induction.

Now let's consider

$$f(x) = \frac{x}{1} + \frac{x^2}{3} - \frac{x^3}{2} + \frac{x^4}{5} + \frac{x^5}{7} - \frac{x^6}{4} + + - - \ldots$$

for $|x| < 1$, where the series is absolutely convergent by comparison with the geometric series. Then we can rearrange this series as follows; write

$$f(x) = f_1(x) + f_2(x) - f_3(x),$$

where

$$f_1(x) = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{4n+1}, \quad f_2(x) = \sum_{n=0}^{\infty} \frac{x^{3n+2}}{4n+3}, \quad f_3(x) = \sum_{n=0}^{\infty} \frac{x^{3n+3}}{2n+2}.$$

Each of these is also absolutely convergent, and we will express them in terms of a power series integrated term by term.

The most basic such example uses the geometric series: if $|y| < 1$, then $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$. Integrating from $y = 0$ to $y = x$ for $|x| < 1$ gives

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots.$$

The geometric series is still valid if we substitute something that's less than 1 in absolute value, but when we integrate the series, other coefficients may emerge. For example, setting $y = t^4$ for $|t| < 1$, we have

$$\frac{1}{1-t^4} = 1 + t^4 + t^8 + t^{12} + \dots$$

This can be integrated term by term from 0 to x, for $|x| < 1$, to get

$$\int_0^x \frac{1}{1-t^4}\, dt = x + \frac{x^5}{5} + \frac{x^9}{9} + \dots.$$

The integral on the left can be evaluated using the standard partial fractions technique: since $1 - t^4 = (1-t)(1+t)(1+t^2)$, for any polynomial $p(t)$ with degree ≤ 3, there exist constants c_j so that

$$\frac{p(t)}{1-t^4} = \frac{c_1}{1-t} + \frac{c_2}{1+t} + \frac{c_3 t + c_4}{1+t^2}.$$

In fact,

$$\frac{1}{1-t^4} = \frac{1/4}{1-t} + \frac{1/4}{1+t} + \frac{1/2}{1+t^2},$$

Thus

$$-\frac{\log(1-x)}{4} + \frac{\log(1+x)}{4} + \frac{1}{2}\arctan x = x + \frac{x^5}{5} + \frac{x^9}{9} + \dots$$

Similarly, we have

$$\frac{t^2}{1-t^4} = t^2 + t^6 + t^{10} + t^{14} + \dots$$

This can be integrated term by term from 0 to x as well. Since

$$\frac{t^2}{1-t^4} = \frac{1/4}{1-t} + \frac{1/4}{1+t} - \frac{1/2}{1+t^2},$$

we have

$$-\frac{\log(1-x)}{4} + \frac{\log(1+x)}{4} - \frac{1}{2}\arctan x = \frac{x^3}{3} + \frac{x^7}{7} + \frac{x^{11}}{11} + \dots$$

Let's return to the original three expressions. If we put $x = y^{4/3}$ (for $0 < x < 1$) in the series for f_1 then $x^{3n+1} = y^{(12n+4)/3} = y^{1/3}y^{4n+1}$, and we can relate the series to the ones described above.

$$f_1(y^{4/3}) = y^{1/3}\sum_{n=0}^{\infty}\frac{y^{4n+1}}{4n+1} = y^{1/3}\left(-\frac{\log(1-y)}{4} + \frac{\log(1+y)}{4} + \frac{1}{2}\arctan y\right),$$

and since $y = x^{3/4}$, we plug in to obtain

$$f_1(x) = x^{1/4}\left(-\frac{\log(1-x^{3/4})}{4} + \frac{\log(1+x^{3/4})}{4} + \frac{1}{2}\arctan x^{3/4}\right).$$

The right hand side is not obviously a power series. That doesn't matter! All I'm saying is that for each $x \in (0,1)$ the value of $f_1(x)$ is given by the function on the right hand side.

Similarly, if we put $x = y^{4/3}$ (for $0 < x < 1$) in the series for f_2, then $x = y^{4/3}$ implies $x^{3n+2} = y^{(12n+8)/3} = y^{-1/3}y^{4n+3}$, hence

$$f_2(y^{4/3}) = y^{-1/3}\sum_{n=0}^{\infty}\frac{y^{4n+3}}{4n+3} = y^{-1/3}\left(-\frac{\log(1-y)}{4} + \frac{\log(1+y)}{4} - \frac{1}{2}\arctan y\right).$$

Substituting back in gives

$$f_2(x) = x^{-1/4}\left(-\frac{\log(1-x^{3/4})}{4} + \frac{\log(1+x^{3/4})}{4} - \frac{1}{2}\arctan x^{3/4}\right).$$

The last piece of the puzzle is the easiest.

$$f_3(x) = \sum_{n=0}^{\infty}\frac{x^{3n+3}}{2n+2} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(x^3)^n}{n} = -\frac{1}{2}\log(1-x^3).$$

Putting this all together, we see that the sum we want is, for $0 < x < 1$,

$$f(x) = \left(\frac{x^{1/4}+x^{-1/4}}{4}\right)\log(1+x^{3/4}) - \left(\frac{x^{1/4}+x^{-1/4}}{4}\right)\log(1-x^{3/4})$$
$$+\left(\frac{x^{1/4}-x^{-1/4}}{2}\right)\arctan x^{3/4} + \frac{1}{2}\log(1-x^3).$$

One last point of interest is to use Abel's Theorem – and this is *much* harder than the analogous part of Homework 11, #10. We have

$$f(1) = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4}\cdots$$

which we know is convergent by Bonus Notes 11 to $\frac{1}{2}\log\frac{4\cdot 2}{1} = \frac{3}{2}\log 2$. And notice that $f_1(1)$, $f_2(1)$ and $f_3(1)$ are all divergent series! But we can use the formulas for $f_j(x)$ for $|x| < 1$ to conclude that $f(1)$ is convergent and so equals $\lim_{x\to 1^-} f(x)$. But what is its value? It's tricky. The easiest way to do it is to let $x = t^4$ and note that as $x \to 1^-$, $t \to 1^-$. We are then looking at the limit as $t \to 1^-$ of

$$\left(\frac{t+1/t}{4}\right)\log(1+t^3) - \left(\frac{t+1/t}{4}\right)\log(1-t^3) + \left(\frac{t-1/t}{2}\right)\arctan t^3 + \frac{1}{2}\log(1-t^{12}).$$

We can pick off a few of these terms. Clearly,

$$\left(\frac{t+1/t}{4}\right)\log(1+t^3) \to \frac{2\log 2}{4}, \qquad \left(\frac{t-1/t}{2}\right)\arctan t^3 \to 0\cdot\tfrac{\pi}{4} = 0.$$

Since $1-t^{12} = (1-t^3)(1+t^3+t^6+t^9)$, the remaining term is

$$\begin{aligned} &-\left(\frac{t+1/t}{4}\right)\log(1-t^3) + \frac{1}{2}\log(1-t^{12}) \\ = &\left(\frac{2-t-1/t}{4}\right)\log(1-t^3) + \frac{1}{2}\log(1+t^3+t^6+t^9) \end{aligned}$$

The second term converges to $\frac{\log 4}{2} = \log 2$. The first is best described by taking $t = 1-u$ and now sending $u \to 0^+$:

$$2-t-\frac{1}{t} = 2-(1-u) - \frac{1}{1^-u} = \frac{-u^2}{1-u}, \quad 1-t^3 = 3u-3u^2+u^3 = u(3-3u+u^2),$$

so

$$\lim_{t\to 1^-}\left(\frac{2-t-1/t}{4}\right)\log(1-t^3) = \lim_{u\to 0^+}\left(\frac{u^2}{1-u}\right)\left(\log u + \log(3-3u+u^2)\right)$$

Finally, since $\lim_{u\to 0^+} u^2\log u = 0$, it follows that this horror just goes to 0 after all, and we've wrestled $\frac{1}{2}\log 2 + \log 2$ out of Abel's Theorem.

If you're bored over Thanksgiving and looking for a challenge, here it is: redo the last few pages for $f(x)$ on $(-1,0)$. I think the easiest way is to let $g(x) = f(-x) = g_1(x) + g_2(x) - g_3(x)$, where

$$g_1(x) = \sum_{n=0}^{\infty}(-1)^{3n+1}\frac{x^{3n+1}}{4n+1}, \quad g_2(x) = \sum_{n=0}^{\infty}(-1)^{3n+2}\frac{x^{3n+2}}{4n+3}, \quad g_3(x) = \sum_{n=0}^{\infty}(-1)^{3n+3}\frac{x^{3n+3}}{2n+2}$$

and note that $(-1)^{3n+1} = (-1)^{3n+3} = (-1)^{n+1}$ and $(-1)^{3n+2} = (-1)^n$. Thanks for staying this far.