Math 347 Homework 10 Solutions Due Wed., Nov. 14, 2001

1. – 24.15, 26.3 (ungraded).

2. – 24.16. Let

$$f_n(x) = \frac{nx}{1+nx^2} = \frac{x}{1/n + x^2}.$$

To find $f(x) := \lim f_n(x)$ for $x \in \mathbf{R}$, we need to separate out $x = 0$ and $x \neq 0$, First, $f_n(0) = 0$ for all n, hence $f(0) = 0$. If $x \neq 0$, then we see from the above that $f(x) = \frac{x}{x^2} = \frac{1}{x}$. Clearly, each f_n is continuous on $\mathbf{R}$, so if $f_n \to f$ uniformly on $[0,1]$ then f would have to be continuous, which it isn't, so the convergence is not uniform. Alternatively, we have that $f_n(0) = f(0)$ for all n, and for $x \neq 0$,

$$|f_n(x) - f(x)| = \left| \frac{nx}{1+nx^2} - \frac{1}{x} \right| = \left| \frac{-1}{x(1+nx^2)} \right| = \frac{1}{x(1+nx^2)}.$$

Thus, since we can ignore $x = 0$ in taking the sup, but think about $x \to 0^+$,

$$\sup\{|f_n(x) - f(x)| : x \in [0,1]\} = \sup\{\frac{1}{x(1+nx^2)} : x \in (0,1]\} = \infty,$$

and so the convergence is not uniform on $[0,1]$. On the other hand, we see that

$$\sup\{|f_n(x) - f(x)| : x \in [1,\infty]\} = \sup\left\{\frac{1}{x(1+nx^2)} : x \in [1,\infty]\right\} = \frac{1}{1+n}.$$

Since $\lim \frac{1}{1+n} = 0$, it follows that the convergence is uniform on $[1,\infty)$. If you prefer the ϵ/N approach, given $\epsilon > 0$, let $N = \frac{1}{\epsilon}$, then if $n > N$ and $x \in [1,\infty)$, then $|f_n(x) - f(x)| = \frac{1}{x(1+nx^2)} \leq \frac{1}{1+n} < \epsilon$.

3. – 25.10. Consider

$$\sum_{n=0}^{\infty} \frac{x^n}{1+x^n}.$$

If $0 \leq x < 1$, then $|1 + x^n| \geq 1$, hence $|\frac{x^n}{1+x^n}| \leq |x|^n$. Since $\sum |x|^n$ converges as usual, it follows that the given series converges by the comparison test. Now suppose a is fixed in $[0,1)$. For $x \in [0,a]$, it follows that $|\frac{x^n}{1+x^n}| \leq a^n$. Applying the Weierstrass M-Test with $M_k = a^k$, we see that the series converges uniformly on $[0,a]$. On the other hand, we have, for $0 \leq x < 1$, $\frac{x^n}{1+x^n} > \frac{x^n}{2}$, and since all terms are positive,

$$\sum_{n=0}^{\infty} \frac{x^n}{1+x^n} \geq \frac{1}{2} \sum_{n=0}^{\infty} x^n = \frac{1}{2(1-x)}.$$

If convergence were uniform on $[0,1)$, then this limiting sum would be continuous, but it's clearly unbounded as $x \to \infty$. The sum is an interesting function in number theory. If you expand it out, the coefficient of x^r in $\sum \frac{x^n}{1+x^n}$ is the number of odd divisors of r minus the number of even divisors of r. So, since the divisors of 12 are 1,2,3,4,6 and 12, the coefficient of x^{12} is $2-4=-2$.

4. – 26.2. I guess I did this in class. For all $x \in (-1,1)$, we have the power series identity

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \dots.$$

We can differentiate this term by term to obtain

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \dots.$$

For each $x \in (-1,1)$, this is a convergent series, and so we can multiply through by the constant x to obtain

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n = x + 2x^2 + 3x^3 + 4x^4 + \dots.$$

It follows from putting $x = \frac{1}{2}$ above that $\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1-1/2)^2} = 2$, and from putting in $x = \pm\frac{1}{3}$, that $\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{1/3}{(1-1/3)^2} = \frac{3}{4}$, and $\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n} = \frac{-1/3}{(1+1/3)^2} = -\frac{3}{16}$.

6. – 26.4. Well, I could just make the observation. What a weird question! Since the series for e^x converges for all x, by putting in $-x^2$, we also get a convergent series: for all x,

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}.$$

We can integrate power series term by term, so that

$$F(x) = \int_0^x e^{-t^2}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1}.$$

There is no obvious way I can think of to simplify $(2n+1)n!$.

7. – 26.6. (In (b), use (a), but don't try to square the series term by term.) Let

$$s(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$c(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

Then differentiating term by term gives

$$s'(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

$$c'(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2n)x^{2n-1}}{(2n)!} = -x + \frac{x^3}{3!} + \dots$$

Since $\frac{2n+1}{(2n+1)!} = \frac{1}{(2n)!}$ and $\frac{2n}{(2n)!} = \frac{1}{(2n-1)!}$, we see that $s'(x) = c(x)$ and $c'(x) = -s(x)$, and it then follows from the Chain Rule that $(c^2 + s^2)' = 2cc' + 2ss' = 2c(-s) + 2sc = 0$ Thus, $c^2 + s^2$ is a constant function. Putting $x = 0$ into the definition of c and s, we see that $c(0) = 1$ and $s(0) = 0$, hence this constant is $1^2 + 0^2 = 1$.

8. – 28.2 a. Let $f(x) = x^3$; to compute the derivative at 2, we observe that

$$\frac{f(x) - f(2)}{x - 2} = \frac{x^3 - 2^3}{x - 2} = x^2 + 2x + 2^2,$$

if $x \neq 2$, and of course is undefined if $x = 2$. Thus,

$$\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} x^2 + 2x + 2^2 = 12,$$

and this is the derivative $f'(2)$. For 28.2d, let $r(x) = \frac{3x+4}{2x-1}$ and we compute the derivative at 1. We have, with some algebraic simplification, for $x \neq 1, \frac{1}{2}$ (where r is undefined) ,

$$\frac{r(x) - r(1)}{x - 1} = \frac{\frac{3x+4}{2x-1} - 7}{x - 1} = \frac{\frac{11-11x}{2x-1}}{x - 1} = -\frac{11}{2x - 1}.$$

It follows that

$$\lim_{x \to 1} \frac{r(x) - r(1)}{x - 1} = -11 = r'(1).$$

9. – 26.8. Some similarity to #5. We have

$$|y| < 1 \implies \sum_{n=0}^{\infty} y^n = \frac{1}{1 - y}$$

Putting in $y = -x^2$ and noting that $|x| < 1$ implies $|-x^2| < 1$, we get

$$\sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1 + x^2}.$$

Applying 26.4, that is to say, integrating term by term from 0 to x, we find

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n + 1} = \arctan x.$$

When $x = 1$, the series on the left is an alternating series of terms which decrease to zero. Thus it is convergent, and by Abel's theorem, we obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \arctan 1 = \tfrac{\pi}{4} \implies \pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right).$$

As $x \to -1$, contrary to what you might think, the same thing occurs: $(-1)^n(-1)^{2n+1} = (-1)^{n+1}$, and the series at the left is just the negative of what we had before. Of course $\arctan(-1) = -\frac{\pi}{4}$, so there's nothing very interesting.

10. – Find a family of polynomials $p_n(x)$ so that

$$f_n(x) = 1 + 3x + 5x^2 + \cdots + (2n+1)x^n = \sum_{k=0}^{n} (2k+1)x^k = \frac{p_n(x)}{(1-x)^2}.$$

[Hints omitted.] You may want to prove the formula you can guess by induction, or any other correct method. I'll guess that

$$p_n(x) = 1 + x - (2n+3)x^{n+1} + (2n+1)x^{n+2}.$$

This is evidently valid for $n = 0$ (note that $1 + x - 3x + x^2 = (1-x)^2$), and so to verify by induction we need evaluate

$$\sum_{k=0}^{n+1} (2k+1)x^k = \frac{1 + x - (2n+3)x^{n+1} + (2n+1)x^{n+2}}{(1-x)^2} + (2n+3)x^{n+1} = \frac{p_{n+1}(x)}{(1-x)^2}$$

Multiplying through by $(1-x)^2$ gives

$$p_{n+1}(x) = 1 + x - (2n+3)x^{n+1} + (2n+1)x^{n+2} + (2n+3)x^{n+1}(1-x)^2 =$$
$$1 + x - (2n+3)x^{n+1} + (2n+1)x^{n+2} + (2n+3)x^{n+1} - (4n+6)x^{n+2} + (2n+3)x^{n+3}$$
$$= 1 + x - (2n+5)x^{n+2} + (2n+3)x^{n+3},$$

which verifies the inductive step for $n \to n+1$.

It follows that if $x \in (-1, 1)$ is fixed, then since

$$\lim_{n\to\infty} nx^n = \lim_{n\to\infty} (n+1)x^{n+1} = \lim_{n\to\infty} (n+2)x^{n+2} = \lim_{n\to\infty} x^n = 0,$$

we have that

$$\lim_{n\to\infty} p_n(x) = \lim_{n\to\infty} \left(1 + x - 2(n+1)x^{n+1} - x^{n+1} + 2(n+2)x^{n+2} - x^{n+2}\right) = 1 + x.$$

Thus, for for $x \in (-1, 1)$,

$$\sum_{k=0}^{\infty} (2k+1)x^k = \frac{1+x}{(1-x)^2}.$$

The point of this problem is that we could also get the answer more simply by manipulating power series, but perhaps we wouldn't believe it:

$$\sum_{k=0}^{\infty} (2k+1)x^k = 2\sum_{k=0}^{\infty} kx^k + \sum_{k=0}^{\infty} x^k = \frac{2x}{(1-x)^2} + \frac{1}{1-x} = \frac{2x + (1-x)}{(1-x)^2} = \frac{1+x}{(1-x)^2}$$