1. – 28.3 (ungraded), 4. – 28.7 (ungraded).

2. – 28.4. We have $f(x) = x^2 \sin \frac{1}{x}$, if $x \neq 0$ and $f(0) = 0$. This is the product of the function x^2, which is differentiable everywhere, and $\sin \frac{1}{x}$, which is the composition of $\sin x$ and $\frac{1}{x}$. We're asked to use that $\sin x$ is differentiable everywhere. If $a \neq 0$, then $\frac{1}{x}$ is differentiable at a, so by 28.4, $\sin \frac{1}{x}$ is differentiable at a and by 28.3, $x^2 \sin \frac{1}{x}$ is differentiable at a. The derivative is computed via the product rule and the chain rule:

$$\left(x^2 \sin \tfrac{1}{x}\right)' = (x^2)' \sin \tfrac{1}{x} + x^2 \left(\sin \tfrac{1}{x}\right)' = 2x \sin \tfrac{1}{x} + x^2 \left(\cos \tfrac{1}{x}\right)\left(-\tfrac{1}{x^2}\right) = 2x \sin \tfrac{1}{x} - \cos \tfrac{1}{x},$$

so if $a \neq 0$, then $f'(a) = 2a \sin \frac{1}{a} - \cos \frac{1}{a}$. On the other hand, to compute $f'(0)$, we use the *definition* of the derivative:

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin \frac{1}{x} - 0}{x} = \lim_{x \to 0} x \sin \tfrac{1}{x},$$

if the limit exists. In fact, we've already seen that this limit exists, and equals 0, so $f'(0) = 0$. If f' were continuous at $x = 0$, then we'd have

$$\lim_{x \to 0} \left(2x \sin \tfrac{1}{x} - \cos \tfrac{1}{x}\right) = 0.$$

However, this limit does not exist. In fact, if we let $x_n = \frac{1}{n\pi}$, then $\sin \frac{1}{x_n} = \sin n\pi = 0$ and $\cos \frac{1}{x_n} = \cos n\pi = (-1)^n$, so that $f(x_n) = (-1)^{n+1}$. Since $x_n \to 0$ and $f'(x_n)$ alternates between -1 and 1, f' cannot continuous at 0, regardless of the value of $f'(0)$.

3. – Repeat 28.4 with $g(x) = x^2 \sin \frac{1}{x^3}$, if $x \neq 0$ and $g(0) = 0$, but without assuming that the same conclusions hold. That is, show that g is differentiable at each $a \neq 0$ and use the usual rules to find $g'(a)$. Then determine whether or not g is differentiable at $x = 0$, and if so, whether g' is continuous at $x = 0$.

The exact same reasoning shows that g is differentiable at $a \neq 0$, and

$$\left(x^2 \sin \tfrac{1}{x^3}\right)' = (x^2)' \sin \tfrac{1}{x^3} + x^2 \left(\sin \tfrac{1}{x^3}\right)' = 2x \sin \tfrac{1}{x^3} + x^2 \left(\cos \tfrac{1}{x^3}\right)\left(-\tfrac{3}{x^4}\right) = 2x \sin \tfrac{1}{x^3} - \tfrac{3}{x^2} \cos \tfrac{1}{x},$$

so $g'(a) = 2a \sin \frac{1}{a^3} - \frac{3}{a^2} \cos \frac{1}{a^3}$. To decide whether g is differentiable at 0, note that

$$\left|\frac{g(x) - g(0)}{x - 0}\right| = \left|x \sin \tfrac{1}{x^3}\right| \leq |x|,$$

and so the limit exists and equals 0; that is, $g'(0) = 0$. To the disappointment of those of you who "knew" that g' must be continuous at 0, it isn't. In fact, using the same x_n as before, we have $g'(\frac{1}{(n\pi)^{1/3}}) = -3\pi^{2/3} n^{2/3} (-1)^n$, which is unbounded as $n \to \infty$; that is, as $x_n \to 0$. Therefore, $\lim_{x \to 0} g'(x)$ does not exist.

5. – 28.14. Not much to show in the first. We have of course

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Just write $x = a + h$, then as $x \to a$, we have $h \to 0$ and $x - a = h$, so that's that. For the second, we assume the first, and then replace h by $-h$, noting that $h \to 0$ implies $-h \to 0$:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h} = \lim_{h \to 0} \frac{f(a) - f(a-h)}{h}.$$

If we add the two limits above, we obtain

$$2f'(a) = \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} + \frac{f(a) - f(a-h)}{h} \right) = \lim_{h \to 0} \frac{f(a+h) - f(a-h)}{h}.$$

Divide by 2 to get the desired limit.

6. – 29.2. By the Mean Value Theorem, and the constantly-assumed knowledge of trigonometric derivatives, we know that if $x \neq y$, then $\frac{\cos x - \cos y}{x - y}$ is equal to the derivative of cos, evaluated at some z between x and y. However, $|-\sin z| \leq 1$ for all real z, hence

$$\left| \frac{\cos x - \cos y}{x - y} \right| = |-\sin z| \leq 1 \implies |\cos x - \cos y| \leq 1 \cdot |x - y|.$$

7. – 29.10. Let $f_2(x) = x^2 \sin \frac{1}{x} + \frac{x}{2}$ for $x \neq 0$ and $f_2(0) = 0$. Then f_2 is the sum of the differentiable function $f(x)$ from 28.4 and the differentiable polynomial $\frac{x}{2}$, and so f_2 is differentiable and $f_2'(x) = f'(x) + \frac{1}{2}$. Further, $f_2'(0) = f'(0) + \frac{1}{2} = \frac{1}{2}$, and for $a \neq 0$,

$$f_2'(a) = 2a \sin \tfrac{1}{a} - \cos \tfrac{1}{a} + \tfrac{1}{2}.$$

Suppose (a, b) is an open interval containing 0, then $b > 0$, hence for n sufficiently large, $0 < x_{2n} < b$. If we take $a = x_{2n} = \frac{1}{2n\pi}$, then

$$f_2'(x_{2n}) = 2x_{2n} \sin 2n\pi - \cos 2n\pi + \tfrac{1}{2} = 0 - 1 + \tfrac{1}{2} = -\tfrac{1}{2}.$$

Since the summands making up f_2' are continuous (except at 0), there exists an open interval (c, d) containing x_{2n} on which f_2' is negative and so on which f is decreasing. Since $(c, d) \cap (a, b)$ is non-empty, we have a part of the interval (a, b) on which f_2 is decreasing, and so there is no open interval containing 0 on which f_2 is increasing. Of course, if we take $a = x_{2n+1}$, then $f_2'(a) = \frac{3}{2}$, so the moral of the story is that f_2 has intervals of increase and decrease alternating as you go to 0.

The comparison with Corollary 29.7(i) is simply this: if $f'(x) > 0$ on (a, b), then f is strictly increasing there, but just knowing f' at a single point a does not tell us about its behavior on an interval containing a. This doesn't contradict linearity either. We know that, for any $\epsilon > 0$, there exists $\delta > 0$ so that $|x| < \delta$ implies

$$\frac{x}{2} - \epsilon|x| < f_2(x) < \frac{x}{2} + \epsilon|x|.$$

This implies for example that $f_2(x) > 0$ for $x > 0$, but does not imply that, if $x > y > 0$, then $f_2(x) > f_2(y) > 0$.

8. – 29.12a. Let $g_1(x) = \tan x - x$. Then for $0 < x < \frac{\pi}{2}$, $g_1'(x) = \sec^2 x - 1 = \tan^2 x \geq 0$, so g_1 is increasing on the interval. Since $g_1(0) = 0$, we see that $g_1(x) \geq 0$, as desired. b. Let $g_2(x) = \frac{x}{\sin x}$. Then for $x \in (0, \frac{\pi}{2})$ by the quotient rule, we have

$$g_2'(x) = \frac{\sin x - x\cos x}{\sin^2 x} = \left(\frac{\cos x}{\sin^2 x}\right)(\tan x - x),$$

which is positive on $(0, \frac{\pi}{2})$ by (a). Since g_2 is continuous as $x \to \frac{\pi}{2}$, we have

$$g_2(x) \leq g_2(\frac{\pi}{2}) \implies \frac{x}{\sin x} \leq \frac{\frac{\pi}{2}}{\sin\frac{\pi}{2}} = \frac{\pi}{2} \implies x \leq \frac{\pi}{2} \cdot \sin x$$

for $x \in (0, \frac{\pi}{2})$. This inequality is also true for $x = 0$ and $x = \frac{\pi}{2}$, so we are done.

9. – 28.8. Let $f(x) = x^2$ if x is rational and $f(x) = 0$ if x is irrational. To prove continuity at $x = 0$, observe that $f(0) = 0$. Suppose $\epsilon > 0$ is given and suppose $|x - 0| < \sqrt{\epsilon}$. If x is rational, then $f(x) = x^2$, so $|f(x) - f(0)| < \epsilon$. If x is irrational, then $f(x) = 0$, so $|f(x) - f(0)| = 0$, In either case, $|f(x) - f(0)| < \epsilon$, so we've satisfied the definition of continuity at 0. We also know that for any real number r, we can find sequences $x_n \to r$ and $y_n \to r$ so that each x_n is rational and each y_n is irrational. In this case, $f(x_n) = x_n^2 \to r^2$ and $f(y_n) = 0 \to 0$. If $r \neq 0$, then $r^2 \neq 0$, and so f cannot be continuous at r. To prove differentiability at $x = 0$, go to the definition, If x is rational, then

$$\frac{f(x) - f(0)}{x - 0} = \frac{x^2 - 0}{x - 0} = x$$

If x is irrational, then

$$\frac{f(x) - f(0)}{x - 0} = \frac{0 - 0}{x - 0} = 0.$$

Thus, in either case, if $|x - 0| < \epsilon$, then

$$\left|\frac{f(x) - f(0)}{x - 0} - 0\right| < \epsilon,$$

so that $f'(0) = 0$. This bizarre example shows that it is possible for highly discontinuous functions to have points of differentiability.

10. [The last hard problem of the semester!] – For $-1 < x < 1$, find a closed form for

$$f(x) = x + \frac{x^2}{3} - \frac{x^3}{2} - \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{7} - \frac{x^7}{6} - \frac{x^8}{8} + \dots$$

where the terms alternate "$++--++--++--\dots$".

We write $f(x) = f_1(x) + f_2(x) - f_3(x) - f_4(x)$, where

$$f_1(x) = \sum_{n=0}^{\infty} \frac{x^{4n+1}}{4n+1}, \quad f_2(x) = \sum_{n=0}^{\infty} \frac{x^{4n+2}}{4n+3}, \quad f_3(x) = \sum_{n=0}^{\infty} \frac{x^{4n+3}}{4n+2}, \quad f_4(x) = \sum_{n=0}^{\infty} \frac{x^{4n+4}}{4n+4}.$$

It seems natural to differentiate $f_1, xf_2, x^{-1}f_3$ and f_4. These series all have a radius of convergence of 1 by the root test, and so it's safe to do this for $x \in (-1, 1)$. We have

$$f_1'(x) = \sum_{n=0}^{\infty} \frac{(4n+1)x^{4n}}{4n+1} = \sum_{n=0}^{\infty} x^{4n} = \frac{1}{1-x^4}.$$

We can integrate term by term, and this is done in Bonus Notes 13 by partial fractions, to obtain

$$f_1(x) = \int_0^x \frac{1}{1-t^4}\, dt = -\frac{\log(1-x)}{4} + \frac{\log(1+x)}{4} + \frac{1}{2}\arctan x.$$

Similarly,

$$f_4'(x) = \sum_{n=0}^{\infty} \frac{(4n+4)x^{4n+3}}{4n+4} = \sum_{n=0}^{\infty} x^{4n+3} = \frac{x^3}{1-x^4},$$

so that (with a simple substitution integral),

$$f_4(x) = \int_0^x \frac{t^3}{1-t^4}\, dt = -\frac{\log(1-x^4)}{4}.$$

We note for later reference that $\log(1-x^4) = \log(1-x) + \log(1+x) + \log(1+x^2)$. The other two need a little more work. Let $g_2(x) = xf_2(x)$. Then

$$g_2(x) = \sum_{n=0}^{\infty} \frac{x^{4n+3}}{4n+3} \implies g_2'(x) = \sum_{n=0}^{\infty} \frac{(4n+3)x^{4n+2}}{4n+3} = \sum_{n=0}^{\infty} x^{4n+2} = \frac{x^2}{1-x^4}.$$

This integral can be done by partial fractions, or as in the Bonus Notes, to give

$$g_2(x) = \int_0^x \frac{t^2}{1-t^4}\, dt = -\frac{\log(1-x)}{4} + \frac{\log(1+x)}{4} - \frac{1}{2}\arctan x.$$

(We divide g_2 by x to get f_2). Finally, let $g_3(x) = x^{-1}f_3(x)$. Then

$$g_3(x) = \sum_{n=0}^{\infty} \frac{x^{4n+2}}{4n+2} \implies g_3'(x) = \sum_{n=0}^{\infty} \frac{(4n+2)x^{4n+1}}{4n+2} = \sum_{n=0}^{\infty} x^{4n+1} = \frac{x}{1-x^4}.$$

Since $\frac{x}{1-x^4} = \frac{x/2}{1-x^2} + \frac{x/2}{1+x^2}$, simple substitutions give

$$g_3(x) = \int_0^x \frac{t}{1-t^4}\, dt = -\frac{\log(1-x^2)}{4} + \frac{\log(1+x^2)}{4},$$

and we multiply by x to get f_3. Putting this all together, and simplifying the logs out, we find that

$$f(x) = f_1(x) + x^{-1}g_2(x) - xg_3(x) - f_4(x) =$$

$$\left(\frac{x-1}{2x}\right)\arctan x + \left(\frac{x^2-1}{4x}\right)\log(1-x) + \left(\frac{(x+1)^2}{4x}\right)\log(1+x) + \left(\frac{1-x}{4}\right)\log(1+x^2). \blacksquare$$

The terms in each of the series

$$1 + \frac{1}{3} - \frac{1}{2} - \frac{1}{4} + \frac{1}{5} + \frac{1}{7} - \frac{1}{6} - \frac{1}{8} + \dots, \qquad -1 + \frac{1}{3} + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{6} - \frac{1}{8} + \dots.$$

go to 0, and the $4n$-th partial sums are the same as the partial sums of the more normal-looking series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{4n-1} - \frac{1}{4n}$ and

$$-1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{4n-1} - \frac{1}{4n} + \dots = -\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \dots + \frac{1}{(4n-1)(4n)},$$

both of which are alternating series which converge, so the desired series converge by the theorem in Bonus Notes 11. By Abel's theorem, their limits are the limits of the $f(x)$ as $x \to 1^-$ and $x \to -1^+$ respectively. We have immediately

$$\lim_{x\to 1^-}\left(\frac{x-1}{2x}\right)\arctan x = 0; \lim_{x\to -1^+}\left(\frac{x-1}{2x}\right)\arctan x = \arctan(-1) = -\frac{\pi}{4}.$$

and

$$\lim_{x\to 1^-}\left(\frac{1-x}{4}\right)\log(1+x^2) = 0; \lim_{x\to -1^+}\left(\frac{1-x}{4}\right)\log(1+x^2) = \frac{\log 2}{2}.$$

For the other two limits, recall that

$$\lim_{u\to 0^+} u\log u = 0 \implies \lim_{x\to 1^-}(1-x)\log(1-x) = 0, \lim_{x\to -1^+}(1+x)\log(1+x) = 0.$$

Thus,

$$\lim_{x\to 1^-}\left(\frac{x^2-1}{4x}\right)\log(1-x) = 0; \lim_{x\to -1^+}\left(\frac{x^2-1}{4x}\right)\log(1-x) = 0.$$

and

$$\lim_{x\to 1^-}\left(\frac{(x+1)^2}{4x}\right)\log(1+x) = \log 2; \lim_{x\to -1^+}\left(\frac{(x+1)^2}{4x}\right)\log(1+x) = 0.$$

Putting this all together, we get

$$1 + \frac{1}{3} - \frac{1}{2} - \frac{1}{4} + \frac{1}{5} + \frac{1}{7} - \frac{1}{6} - \frac{1}{8} + \dots = \log 2$$

$$-1 + \frac{1}{3} + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{6} - \frac{1}{8} + \dots = \frac{\log 2}{2} - \frac{\pi}{4}$$

The first sum can be found by recognizing that the $4n$-th partial sums are also the $4n$-th partial sum of $\sum \frac{(-1)^{n+1}}{n}$.