Math 347 "Homework 12" Solutions "Due" Fri., Dec. 7, 2001

1., 2. – 29.9, 29.13 (ungraded). Can't start writing solutions to these now!

3. – 29.14. The inequality $x \le f(x) \le 2x$ is trivial for $x = 0$. If $x > 0$, then by the Mean Value Theorem, there exists y, $0 < y < x$, so that

$$\frac{f(x) - f(0)}{x - 0} = f'(y) \implies \frac{f(x)}{x} = f'(y).$$

Since $1 \le f'(y) \le 2$, it follows that $1 \le \frac{f(x)}{x} \le 2$, so $x \le f(x) \le 2x$.

4. – 32.2. If $f(x) = x$ for rational x and $f(x) = 0$ for irrational x, then for any interval $S \subseteq [0, b]$, $M(f, S) = \sup S$ and $m(f, S) = 0$. (This is because any interval contains irrational numbers and also rational numbers arbitrarily close to any endpoint.) Given a partition $P = \{0 < t_1 < \cdots < t_n = b\}$, we then have the Upper and Lower Darboux Sums equal to

$$U(f, P) = \sum_{k=1}^{n} t_k(t_k - t_{k-1}), \qquad L(f, P) = 0.$$

As we saw in bonus handout 14,

$$U(f) = \inf_P U(f, P) = \frac{1}{2}b^2, \qquad L(f) = \sup_P L(f, P) = 0.$$

(Here, the Upper Darboux sums are the same as those for $f(x) = x$.). Since $\frac{1}{2}b^2 \ne 0$ for $b > 0$, this means that f is not integrable.

5. – 33.4. One such example is if $f(x) = 1$ if x is rational and $f(x) = -1$ if x is irrational. It is easy to see that $U(f, P) = 1$ and $L(f, P) = -1$ for any partition P of $[0, 1]$. Thus, $U(f) = 1$ and $L(f) = -1$ and f is not integrable. However, $|f|$ is just the constant function 1, which is integrable.

6. – 33.8. I guess this needs 33.7 as well. Suppose f is a bounded function on $[a, b]$ and suppose $|f(x)| \le B$ for all x. If S is any subset of $[a, b]$ and $q, r \in S$, then

$$|f^2(q) - f^2(r)| = |f(q) - f(r)| \cdot |f(q) + f(r)| \le 2B \cdot |f(q) - f(r)| \le 2B(M(f, S) - m(f, S)).$$

It follows that $M(f^2, S) - m(f^2, S) \le 2B(M(f, S) - m(f, S))$. If we now fix any particular partition P, we find that

$$(*) \quad \begin{aligned} U(f^2, P) - L(f^2, P) &= \sum_{k=1}^{n} \left(M(f^2, [t_{k-1}, t_k]) - m(f^2, [t_{k-1}, t_k])\right)(t_k - t_{k-1}) \\ &\le 2B \sum_{k=1}^{n} \left(M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])\right)(t_k - t_{k-1}) = 2B(U(f, P) - L(f, P)). \end{aligned}$$

Suppose f is integrable and $\epsilon > 0$ is given. Then by 32.7, there exists δ so that if P is a partition with mesh$(P) < \delta$, then $U(f,P) - L(f,P) < \frac{\epsilon}{2B}$. But (*) shows that if P is such a partition, then $U(f^2,P) - L(f^2,P) < \epsilon$, so that f^2 is integrable. We remark that this requires f to be bounded in the generalized definition of integration: if $f(x) = x^{-1/2}$ on $[0,1]$, then f is integrable in any reasonable sense, but f^2 is not.

Now we go to 33.8. We have to assume boundedness, because otherwise we could take $f = g = x^{-1/2}$. If f and g are bounded and integrable, then so are $f+g$ and $f-g$ and hence so are $(f+g)^2$ and $(f-g)^2$ by the above, and hence so is $fg = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right)$.

If f and g are integrable, then so is $|f-g|$ by 33.5, and hence so are

$$\max(f,g) = \frac{f+g}{2} + \frac{|f-g|}{2}, \qquad \min(f,g) = \frac{f+g}{2} - \frac{|f-g|}{2}.$$

7. – 34.2. Let $F(x) = \int_0^x e^{t^2}\,dt$. Then $F(0) = 0$ and by the Fundamental Theorem of Calculus,

$$\lim_{x\to 0} \frac{F(x)}{x} = \lim_{x\to 0} \frac{F(x) - F(0)}{x - 0} = F'(0) = e^{0^2} = 1.$$

Similarly, the second limit can be written as

$$\lim_{h\to 0} \frac{F(3+h) - F(3)}{h} = F'(3) = e^{3^2} = e^9.$$

8. – 34.6. Suppose f is a continuous function on $\mathbf{R}$ and let $F(x) = \int_0^x f(t)\,dt$. We know from 34.3 that F is a differentiable function and $F'(x) = f(x)$. Now we are given

$$G(x) = \int_0^{\sin x} f(t)\,dt = F(\sin(x)),$$

which is a composition of two differentiable functions. Then G is also differentiable on $\mathbf{R}$ and by the Chain Rule,

$$G'(x) = F'(\sin x) \cdot (\sin x)' = (\cos x) f(\sin x).$$

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9. – 29.18. Here f is differentiable on $\mathbf{R}$ and

$$\alpha = \sup\{|f'(x)| : x \in \mathbf{R}\} < 1.$$

(This is a stronger statement than $|f'(x)| < 1$; it's the statement that there exists a positive number $c = 1 - \alpha$ so that $|f'(x)| \le 1 - c < 1$.) I am renaming a in the book as α to avoid symbolic confusion in the Mean Value Theorem.

We now define a sequence recursively by $s_0 \in \mathbf{R}$, and $s_n = f(s_{n-1})$ for $n \ge 1$. We want to show that (s_n) is convergent. The Mean Value Theorem is helpful. Let $a = s_n$

and $b = s_{n-1}$; then $f(a) = s_{n+1}$ and $f(b) = s_n$. (The fact that $a = f(b)$ is true, but will be suppressed below.) The Mean Value Theorem implies that there exists c between a and b so that

$$\left|\frac{f(a) - f(b)}{a - b}\right| = |f'(c)| \implies \left|\frac{s_{n+1} - s_n}{s_n - s_{n-1}}\right| \leq \alpha \implies |s_{n+1} - s_n| \leq \alpha |s_n - s_{n-1}|.$$

Iterating this, we obtain the inequality $|s_{n+1} - s_n| \leq \alpha^n |s_1 - s_0|$. We want to show that (s_n) is convergent, and this basically comes from looking at the telescoping series $\sum_k (s_{k+1} - s_k)$. Suppose $\epsilon > 0$ is given. If $m > n \geq N$, then

$$\begin{aligned} s_m - s_n = \sum_{k=n}^{m-1} s_{k+1} - s_k &\implies |s_m - s_n| = \sum_{k=n}^{m-1} |s_{k+1} - s_k| \leq |s_1 - s_0| \sum_{k=n}^{m-1} \alpha^k \\ &= |s_1 - s_0| \cdot \frac{\alpha^n - \alpha^m}{1 - \alpha} < \frac{|s_1 - s_0| \cdot \alpha^n}{1 - \alpha} < \frac{\alpha^N |s_1 - s_0|}{1 - \alpha} \end{aligned}$$

For N sufficiently large, the upper bound above can be made smaller than any fixed $\epsilon > 0$, hence (s_n) is Cauchy, and thus is convergent.

Any function f for which $|f(x) - f(y)| < |x - y|$ is called a *contraction mapping*, and these can be found in quite general settings, like metric spaces.

10. – 34.10. More generally, but with the same proof, show that if f is a strictly increasing continuous function which maps the interval $[0, a]$ onto the interval $[0, b]$ $(a, b > 0, f(a) = b)$, then

$$\int_0^a f(x)\ dx + \int_0^b f^{-1}(x)\ dx = ab.$$

When done correctly, the proof requires no words. See picture. Tilt your head to one side if necessary.