1. (not graded)– 7.3. The "odd ones" have the answers in the back. The answers for the "even" ones are: (b) converges to 1, (d) converges to 1, (f) converges to 1, (h) does not converge, (j) converges to $\frac{7}{2}$, (l) does not converge, (n) does not converge, (p) converges to 2, (r) converges to 1, (t) converges to 0.

I will be happy to discuss any of these in class upon request.

2. – 7.4. There are infinitely many such examples. I'll give a couple of examples, and report to the class with others.

(a) Let $x_n = \frac{1}{n} \cdot \sqrt{2}$. Then each x_n is irrational, because $\sqrt{2}$ is irrational. But clearly, $\lim x_n = 0$, which is rational.

(b) The simplest version might be the sequence where y_n is the decimal representation of $\sqrt{2}$ to n places, so that $y_1 = 1.4$, $y_2 = 1.41$, $y_3 = 1.414$, etc. Then each y_n is rational, and $|y_n - \sqrt{2}| < 10^{-n}$, which implies that $\lim y_n = \sqrt{2}$, which is irrational. This can be written more formally by writing $y_n = 10^{-n}\lfloor 10^n\sqrt{2}\rfloor$.

3. – 8.2 ad. A formal proof is requested of these.

(a) Let $s_n = \frac{n}{n^2+1}$. Then we know that $s_n \to 0$. How to prove it? Well clearly, $s_n > 0$, and $|s_n| < \epsilon$ can be written as follows

$$\frac{n}{n^2+1} < \epsilon \iff \frac{n}{\epsilon} < n^2 + 1 \iff 0 < n^2 - \frac{n}{\epsilon} + 1.$$

By the quadratic formula, this is true unless n is between the two roots of the equation $n^2 - \epsilon^{-1}n + 1 = 0$. Thus, when $n > \frac{\epsilon^{-1}+\sqrt{\epsilon^{-2}-4}}{2}$, $\frac{n}{n^2+1} < \epsilon$ and we can take this ugly expression for $N = N(\epsilon)$, so that if $n > N(\epsilon)$, then $|s_n - 0| < \epsilon$. A less painful way to do this is to make the estimate

$$\frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n},$$

so that, if $n > \epsilon^{-1}$, then $|s_n - 0| < \epsilon$. Observe that the ugly expression above is pretty darned close to ϵ^{-1} in any case.

(d) Let $s_n = \frac{2n+4}{5n+2}$. Then it's pretty obvious from our calculus days that $s_n \to \frac{2}{5}$. The rest of this is similar to Example 2, p.38. We have

$$\left|\frac{2n+4}{5n+2} - \frac{2}{5}\right| = \frac{16}{25n+10},$$

and, as we've done before,

$$\frac{16}{25n+10} < \epsilon \iff 16 < \epsilon(25n+10) \iff \frac{16}{25\epsilon} - \frac{2}{5} < n.$$

Taking $N(\epsilon)$ to be the expression above, we see that $n > N(\epsilon)$ implies that $|s_n - s| < \epsilon$. This is true for any $\epsilon > 0$ and therefore the limit is proved.

4. – (not graded) 8.5. See the proof in the back. I forgot in class that I'd assigned it as a problem when I was using it in class examples.

5. – 8.10. Let $\lim s_n = s$ and assume that $s > a$. Let $\epsilon = s - a$. (Actually, anything positive and $\le s - a$ will work.) By the definition of convergence, there exists N so that for $n > N$, we have $|s - s_n| < \epsilon$. But if this equation holds, then $s_n > s - |\epsilon|$ by the triangle inequality, and by our *choice* of ϵ, we have $s_n > s - (s - a) = a$. This is a proof that can be mapped out in advance by drawing a picture or two.

6. – 9.2. Using the limit theorems, if $\lim x_n = 3$, $\lim y_n = 7$ and $y_n \neq 0$, then (a) $\lim(x_n + y_n) = \lim x_n + \lim y_n = 3 + 7 = 10$, and (b)

$$\lim\left(\frac{3y_n - x_n}{y_n^2}\right) = \frac{\lim(3y_n - x_n)}{\lim y_n \cdot \lim y_n} = \frac{3 \cdot 7 - 3}{7^2} = \frac{18}{49}.$$

7. – 9.6. Here $x_1 = 1$ and $x_{n+1} = 3x_n^2$.

(a) Suppose $a = \lim x_n$. Then $a = \lim x_{n+1}$ as well (see bonus theorem at end), and so $a = \lim 3x_n^2 = 3a^2$. This implies that $a = 0$ or $a = \frac{1}{3}$.

(b) However, we see that $x_2 = 3$, $x_3 = 27$, etc; it's easy to see that x_n increases without bound so that $\lim x_n = \infty$. More formally, to get an exact formula for x_n (which wasn't asked!), make the substitution $x_n = 3^{y_n}$, so that $3^{y_{n+1}} = 3 * 3^{2y_n}$, hence $y_{n+1} = 2y_n + 1$ and $y_1 = 0$. Many people in this class know how to solve this equation, and a little experimentation shows that $y_n = 2^n - 1$, which is easy to prove by induction. Thus

$$x_n = 3^{2^n - 1}.$$

This sequence grows very quickly. For example, $x_{10} = 3^{1023} \approx 1.2446 \times 10^{488}$.

(c) There is no contradiction. If $\lim x_n = \infty$, then x_n does not converge, and there is nothing that can be done algebraically with this equation. An alternative intuitive answer is that $a = \infty$ is also a solution to $a = 2a^2$. This isn't quite right, because ∞ is not a number, just a marker of unbounded behavior.

8. The sequence $b_n = \sin(\frac{n\pi}{6})$ is periodic, with a cycle of 12, taking in order the values

$$\left\{\frac{1}{2},\quad \frac{\sqrt{3}}{2},\quad 1,\quad \frac{\sqrt{3}}{2},\quad \frac{1}{2},\quad 0,\quad -\frac{1}{2},\quad -\frac{\sqrt{3}}{2},\quad -1,\quad -\frac{\sqrt{3}}{2},\quad 0.\right\}$$

Thus, the first few terms of $a_n = \frac{b_n}{n}$ are

$$a_1 = \frac{1}{2},\quad a_2 = \frac{\sqrt{3}}{4},\quad a_3 = \frac{1}{3},\quad a_4 = \frac{\sqrt{3}}{8},\quad a_5 = \frac{1}{10},\quad a_6 = 0,$$
$$a_7 = -\frac{1}{14},\quad a_8 = -\frac{\sqrt{3}}{16},\quad a_9 = -\frac{1}{9},\quad a_{10} = -\frac{\sqrt{3}}{20},\quad a_{11} = -\frac{1}{22},\quad a_{12} = 0, \ldots$$

The first thing to say is that $|a_n| \le \frac{1}{n}$, so $\lim a_n = 0$. To find a supremum, we'd like to find the largest member of the sequence and prove it. Eyeballing things, $a_1 = \frac{1}{2}$ is a good candidate to be a supremum. Indeed, if $n \ge 2$, then

$$|a_n| = \left|\frac{\sin(\frac{n\pi}{6})}{n}\right| \le \frac{1}{n} \le \frac{1}{2}.$$

It follows that $\frac{1}{2}$ is indeed an upper bound, and it is obviously the least upper bound, or the supremum.

The infimum is found in a similar way. Some experimentation shows that $a_9 = -\frac{1}{9}$ is the smallest element in the table above, so $a_n \ge a_9$ for $n \le 12$. And $|a_n| \le \frac{1}{n}$, so $|a_n| \le \frac{1}{13}$ for $n \ge 13$, hence $a_n \ge -\frac{1}{13} > -\frac{1}{9}$. Thus, $a_9 = -\frac{1}{9}$ is the infimum.

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9. I will prove a slightly stronger result. Suppose N is a natural number and $a < b$ are real numbers. I will show that there exist N rational numbers $r_j = \frac{m_j}{n_j}$ $(m_j, n_j \in \mathbf{Z})$ so that

$$a < r_1 < \cdots < r_N < b.$$

Indeed, all we need to do is define $c_j = a + \frac{j}{N}(b-a)$. Then $c_0 = a$, $c_0 < c_1 < \cdots < c_{N-1} < c_N = b$. Since $b - a > 0$, it is also true that $\frac{b-1}{N} > 0$; that is why $c_j < c_{j+1}$. By Theorem 4.7, there exists a rational number r_j so that $c_j < r_j < c_{j+1}$, and these will do the trick.

10. Prove that $A \times B$ is a countable set if both A and B are countable sets. The fastest proof I know is to let

$$A_i = \{(a_i, b_1), (a_i, b_2), (a_i, b_3), \dots\}.$$

Then the elements in A_i can be counted along with B, and A_i is a countable set, and $A \times B = \cup_i A_i$ by the definition of the cross product, since each element $(a_i, b_j) \in A \times B$ is in A_i. Thus, $A \times B$ is the countable union of countable sets, and so is countable. These sets are also disjoint, though that's not necessary. Diagonalizing the set as we did in the handout gives, as the first 6 elements,

$$(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_1, b_3), (a_3, b_2), (a_3, b_1),$$

but other valid diagonalizations are acceptable of course.

Bonus Theorem

Theorem. Suppose r is a fixed integer and $\lim s_n = s$. If $t_n = s_{n+r}$, then $\lim t_n = s$.
Proof. Given $\epsilon > 0$, there exists N so that $n > N \implies |s_n - s| < \epsilon$. But then $n > N - r$ implies that $n + r > N$, so that $|t_n - s| = |s_{n+r} - s| < \epsilon$.