1. – 9.12. Suppose $s_n \neq 0$ and $L = \lim \left|\frac{s_{n+1}}{s_n}\right|$ exists.

a. If $L < 1$, prove that $\lim s_n = 0$. Following the hint, choose a so that $L < a < 1$. (If you need to be specific, let $a = \frac{1+L}{2}$.) Then, taking $\epsilon = a - L > 0$, there exists N so that

$$n \geq N \implies \left|\left|\frac{s_{n+1}}{s_n}\right| - L\right| < \epsilon \implies \left|\frac{s_{n+1}}{s_n}\right| < L + \epsilon = a.$$

(This is *not quite* the way we usually write this, but $n \geq N$ is the same as $n > N - 1$.) Thus for $n \geq N$, $|s_{n+1}| < a|s_n|$, so $|s_{n+2}| < a|s_{n+1}| < a^2|s_n|$, etc. It is an easy induction to show that $|s_{N+k}| < a^k|s_N|$. Rewriting $n = m - N$ gives

$$m \geq N \implies |s_m| < a^m|s_N| \implies -a^m|s_N| < s_m < a^m|s_N|.$$

Since s_N is a constant and $a < 1$, $a_n \to \infty$, so $\lim s_m = 0$.

b. If $L > 1$, prove that $\lim |s_n| = \infty$. Again, following the hint, let $t_n = |s_n|^{-1}$, which is defined because $s_n \neq 0$. Then,

$$\lim\left(\frac{t_{n+1}}{t_n}\right) = \lim\left(\frac{\frac{1}{|s_{n+1}|}}{\frac{1}{|s_n|}}\right) = \lim\left(\frac{|s_{n+1}|}{|s_n|}\right)^{-1} = L^{-1} < 1.$$

Now $t_n > 0$, and by part (a), we now know that $\lim t_n = 0$. Hence by Theorem 9.10, $\lim t_n^{-1} = \infty$, that is, $\lim s_n = \infty$.

2. – 10.1 (ungraded). Answers in back. I'll talk about any in class that you want to.

3. – 10.6 (In b., either prove the assertion or give a counterexample.)

(a) Suppose (s_n) is a sequence so that $|s_{n+1} - s_n| < 2^{-n}$ for all n. We wish to prove that (s_n) is a Cauchy sequence (and so is convergent). Some discussion first. Suppose $m > n > N$ and $m = n + k$, to be specific. Then we can go from s_n to s_m in "steps":

$$s_m - s_n = s_{n+k} - s_n = (s_{n+k} - s_{n+k-1}) + (s_{n+k-1} - s_{n+k-2}) + \cdots + (s_{n+1} - s_n).$$

It follows from the triangle inequality that

$$|s_m - s_n| \leq |s_{n+k} - s_{n+k-1}| + \cdots + |s_{n+1} - s_n| < \frac{1}{2^{n+k-1}} + \cdots + \frac{1}{2^n}.$$

We don't lose too much by taking the rest of the geometric series, so,

$$(*) \qquad |s_m - s_n| \leq \sum_{r=n}^{n+k-1} \frac{1}{2^r} < \sum_{r=n}^{\infty} \frac{1}{2^r} = \frac{1}{2^{n-1}} < \frac{1}{2^{N+1-1}} = \frac{1}{2^N}.$$

Here, I've used $n > N \implies n \geq N + 1$. If $m = n$, then (*) is trivial, and if $m < n$, the same argument applies, so (*) holds whenever $m, n > N$. Suppose now $\epsilon > 0$ is given. There exists N so that $\epsilon > \frac{1}{2^N}$. Then for $m, n > N$, (*) implies that

$$|s_m - s_n| < \frac{1}{2^N} < \epsilon,$$

and (s_n) is a Cauchy sequence. A concrete example of such a sequence is the following. A random process chooses $\epsilon_n \in [-1, 1]$. Then the infinite series $\sum_{n=1}^{\infty} \frac{\epsilon_n}{2^n}$ always converges.

(b) The result is false. For example, let

$$s_n = \sum_{k=1}^{n} \frac{1}{k} \implies s_{n+1} - s_n = \frac{1}{n+1} < \frac{1}{n}.$$

Then, as we know from calculus, $\lim s_n = \infty$. If (s_n) were Cauchy, then it would be convergent, but it isn't. A direct proof of non-Cauchy-ness is familiar:

$$s_{2n} - s_n = \frac{1}{n+1} + \cdots + \frac{1}{2n} > \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{n}{2n} = \frac{1}{2},$$

so if $\epsilon < \frac{1}{2}$, then there does not exist N so that $n, m > N \implies |s_n - s_m| < \epsilon$.

4. – 10.9 (ungraded) Answers in back. Ask questions if you want.

5. – 10.12. An interesting pedagogical question. You can never tell anybody that how they feel is "wrong", so I can't really grade (b) sensibly. We have $t_1 = 1$ and

$$t_{n+1} = \left(1 - \frac{1}{(n+1)^2}\right) t_n = \frac{n(n+2)}{(n+1)^2} \cdot t_n.$$

It is easy to see that $t_n > 0$ for all n and $t_n > t_{n+1}$. Thus (t_n) is a bounded monotone decreasing sequence, and so it has a limit. The previous tricks fail, because if $t = \lim t_n$, then the recurrence tells us that $t = 1 \cdot t$, which isn't very helpful. The key is induction. We have $t_1 = 1 = \frac{1+1}{2}$, and assuming that $t_n = \frac{n+1}{2n}$, we have

$$t_{n+1} = \frac{n(n+2)}{(n+1)^2} \cdot \frac{n+1}{2n} = \frac{n+2}{2(n+1)},$$

establishing the induction. Then, $\lim t_n = \lim \frac{n}{2n+1} = \lim \frac{1}{2+1/n} = \frac{1}{2}$.

6. – 11.2 (do a_n, d_n.)

Let $a_n = (-1)^n$, then as noted in class, (a) $(a_{2n}) = (1)$ is a monotone subsequence, (b) the subsequential limits are -1 and 1, (c) $\limsup a_n = 1$ and $\liminf a_n = -1$, (d) the sequence does not converge, (e) the sequence is bounded.

Let $d_n = \frac{6n+4}{7n+3}$. Then as we've seen $d_n \to \frac{6}{7}$ and

$$d_{n+1} - d_n = \frac{6n+10}{7n+10} - \frac{6n+4}{7n+3} = \frac{-10}{(7n+3)(7n+10)} < 0.$$

It follows that (a) any subsequence is monotone, (b) the only subsequential limit is $\frac{6}{7}$, (c) $\limsup d_n = \liminf d_n = -\frac{6}{7}$, (d) the sequence is convergent and (e) the sequence is bounded.

7. – 11.4 (do x_n, y_n.)

We have $x_n = 5^{(-1)^n}$, so x_n alternates between 5 and $\frac{1}{5}$. Therefore, (a) the subsequence $(x_{2n}) = 5$ is monotone, (b),the subsequential limits are 5 and $\frac{1}{5}$, (c) $\limsup x_n = 5, \liminf x_n = \frac{1}{5}$, (d) the sequence is not convergent, (e) the sequence is bounded.

I think I meant z_n, because $y_n = 1+(-1)^n$ alternates between 0 and 2 and the answers are the same as for x_n, with "5" replaced by "2" and "$\frac{1}{5}$" replaced by "0". Sorry.

8. – 11.10. The sequence (s_n) as given is $1, \frac{1}{2}, 1, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, \cdots$. It's clear that for any integer n, the number $\frac{1}{n}$ appears infinitely many times in the sequence, and so, by selecting its appearances, we can find a subsequence converging to $\frac{1}{n}$. Further, we can find the subsequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$, which converges to 0. I claim that these are the only subsequential limits. It's clear that if $x > 1$ or $x < 0$, then for sufficiently small ϵ, $|x - s_n| > \epsilon$ for every n, hence x can't be a subsequential limit. The remaining case is $x \in (0,1)$, but $\frac{1}{x} \notin \mathbf{N}$. Let $\delta = \min_n |x - \frac{1}{n}|$. Then for $0 < \epsilon < \delta$, we have $|x - s_n| > \epsilon$ for every n. If you want an exact value for δ, it is

$$\min \left\{ \left| \frac{1}{\lfloor 1/x \rfloor} - x \right|, \left| \frac{1}{\lceil 1/x \rceil} - x \right| \right\}.$$

It's pretty clear that for all N, $\sup\{s_n : n > N\} = 1$ and $\inf\{s_n : n > N\} = 0$, so $\limsup s_n = 1$ and $\liminf s_n = 0$.

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9. Let $x_1 = \sqrt{3}$, $x_2 = \sqrt{3 + \sqrt{3}}$, $x_3 = \sqrt{3 + \sqrt{3 + \sqrt{3}}}$, etc. Using the techniques from the discussion of the similar example (with 2 instead of 3), prove that $\lim x_n$ exists, and compute it.

We have $x_n^2 = 3 + x_{n-1}$. If $\lim x_n = x$, then $x^2 = 3 + x$, so $x^2 - x - 3 = 0$ and $x = \frac{1 \pm \sqrt{13}}{2}$. Since $x > 0$, we must have $x = \frac{1+\sqrt{13}}{2} \approx 2.302$. If this is all you do, then you don't get full credit, because you still have to prove that this limit exists! For simplicity, I'll call this limit α.

Mimicking the proof of the handout, I will first show that $x_{n+1} > x_n$ by induction. This is true for $n = 1$ directly, and assuming that this inequality holds, then

$$x_{n+2}^2 = 3 + x_{n+1} > 3 + x_n = x_{n+1}^2,$$

hence $x_{n+2} > x_{n+1}$. Next, I want to show that (x_n) is bounded above. An easy induction shows that $x_n < \alpha$: clearly, $x_1 < \alpha$, and if $x_n < \alpha$, then $x_{n+1} < \sqrt{3 + \alpha} = \sqrt{\alpha^2} = \alpha$.

At this point, it is correct to note that (x_n) is a bounded monotone increasing function, and so it has a limit, and argue as I did above that the limit is equal to α. For the sake of those who followed by previous proof exactly, let $y_n = \alpha - x_n$, so that $x_n = \alpha - y_n$ and $y_n < y_1 = \alpha - \sqrt{3} \approx .571$. Then $y_n > y_n^2$, and since $\alpha^2 = 3 + \alpha$,

$$x_n^2 = 3 + x_{n-1} \implies \alpha^2 - 2\alpha y_n + y_n^2 = 3 + \alpha - y_{n-1}$$
$$\implies y_{n-1} = 2\alpha y_n - y_n^2 > 2\alpha y_n - y_n = (2\alpha - 1)y_n = \sqrt{13} \cdot y_n.$$

Thus, $\frac{y_n}{y_{n-1}} < \frac{1}{\sqrt{13}}$, and as before, $y_n \to 0$, so $x_n \to \alpha$.

10. Let $f(x) = 2x^2 - 1$. For $a \in \mathbf{R}$, define a sequence (s_n), $n \geq 0$, by $s_0 = a$ and $s_{n+1} = f(s_n)$, for $n \geq 0$. Thus, $s_1 = 2a^2 - 1$, $s_2 = 2(2a^2-1)^2 - 1 = 8a^4 - 8a^2 + 1$, etc.

a. Suppose (s_n) is a convergent sequence and $s_n \to s$. Determine the two possible values of s. Well, $s_n \to s$ implies $s_{n+1} \to s$, as we've seen, so $s = 2s^2 - 1$, and $s = 1$ or $s = -\frac{1}{2}$.

b. Suppose $a > 1$. Show that $s_n \to \infty$. (Hint: determine, with proof, a constant $\lambda > 1$ so that, if $a > 1$, then $f(a) - 1 > \lambda(a-1)$.) Observe that

$$f(1+b) = 2(1+b)^2 - 1 = 2 + 4b + 2b^2 - 1 = 1 + 4b + b^2 > 1 + 4b.$$

Thus, $\lambda = 4$ will work. So, if $s_1 = 1+b$, then $s_2 > 1+4b$, and by induction, $s_n > 1+4^{n-1}b$. If $b > 0$, this implies that $\lim s_n = \infty$. (By the way, this inequality holds when $b < 0$, but it doesn't tell you anything very interesting.)

c. Suppose $|a| \leq 1$. Prove that (s_n) is a bounded sequence. (Hint: look at the image of the interval $[-1,1]$ under f.) If $|a| \leq 1$, then $0 \leq 2a^2 \leq 2$, so $-1 \leq 2a^2 - 1 \leq 1$, hence $|s_n| \leq 1 \implies |s_{n+1}| \leq 1$. It follows that if $s_1 \in [-1,1]$, then $s_n \in [-1,1]$ for all n.

d. Prove that $f(\cos(\theta)) = \cos(2\theta)$, and use this formula to give an explicit expression for s_n, when $s_0 = a = \cos(\theta)$. This is immediate from the trig identity $\cos 2\alpha = 2\cos^2\alpha - 1$. An easy and omitted induction implies that $s_n = \cos(2^n\theta)$. (Also, if $s_0 = \cosh u$, then $s_n = \cosh 2^n u$.)

e. Show that there are countably many different values of $s_0 = a$ for which (s_n) is a convergent sequence. Suppose

$$s_0 = a = \cos\left(\frac{r\pi}{2^m}\right), \qquad 0 \leq r \leq 2^m, \quad r \in \mathbf{N}$$

Then for $n > m$, $2^{n-m}r\pi$ is an integer multiple of 2π, hence $s_n = 1$, and so $s_n \to 1$. Clearly, there are infinitely many different values of a given above, and since $\frac{r}{2^m} \in \mathbf{Q}$, it is a countable number. (Since $\cos\frac{2\pi}{3} = \cos\frac{4\pi}{3} = -\frac{1}{2}$, we can get infinitely many values of a so that the sequence becomes constant at $-\frac{1}{2}$ by taking $a = \cos\left(\frac{r\pi}{3\cdot 2^m}\right)$, where $3 \not| \, r$.)

It is not hard to show that if $a = \cos\left(\frac{a}{2^m b}\pi\right)$, where a and b are relatively prime and b is odd, then (s_n) will (eventually) be periodic, with period at most $b-1$. Also, if $a = \cos(\beta\pi)$, where β is irrational, then (s_n) is not only not convergent or periodic, but doesn't ever take the same value twice. This illustrates the very sensitive dependence on initial conditions. Finally, for every integer n, $\cos(n \arccos x)$ is a polynomial in x, whose leading term is $2^{n-1}x^n$. These are called the Chebyshev polynomials, and some mathematicians have spent their lives studying them.