1., 2, – (ungraded) 12.1, 12.3 (see back of book, ask in class if you have questions).

3. – 12.4. Suppose (s_n) and (t_n) are bounded sequences. Prove that $\limsup(s_n + t_n) \le \limsup s_n + \limsup t_n$.

First, using the book's hint. Let $a_N = \sup\{s_n : n > N\}$, $b_N = \sup\{t_n : n > N\}$ and $c_N = \sup\{s_n + t_n : n > N\}$. Then for all $n > N$, we have $a_N \ge s_n$ and $b_N \ge t_n$, since these are upper bounds of the associated sets. This implies that for all $n > N$, $a_N + b_N \ge s_n + t_n$, and since c_N is the *least* number with this property, $a_N + b_N \ge c_N$. We know that these are bounded sequences, so $\lim a_N = a = \limsup s_n$, $\lim b_N = b = \limsup t_n$ and $\lim c_N = c = \limsup(s_n + t_n)$ all exist. Since $a_N + b_N - c_N \ge 0$ for all N and $\lim a_N + b_N - c_N = a + b - c$ exists, we have $a + b - c \ge 0$; that is, $a + b \ge c$.

As an alternative proof, pick $\epsilon > 0$. Then there exists N_1 so that, for $n > N_1$, $s_n \le \limsup s_n + \epsilon/2$ and there exists N_2 so that, for $n > N_2$, $t_n \le \limsup t_n + \epsilon/2$. Hence, for $n > \max\{N_1, N_2\}$, $s_n + t_n \le \limsup s_n + \limsup t_n + \epsilon$, and so $\limsup(s_n + t_n) \le \limsup s_n + \limsup t_n + \epsilon$. Since this is true for every $\epsilon > 0$, we are done.

4. – 12.6. Suppose (s_n) is a bounded sequence and $k \ge 0$ is a nonnegative real number.

a. Prove that $\limsup ks_n = k \limsup s_n$. Well, let $v_N = \sup\{s_n : n > N\}$. Let $w_N = \sup\{ks_n : n > N\}$. I claim that $w_N = kv_N$. This isn't too hard to see: if $k = 0$, w_N is the supremum of a set of 0's. If $k > 0$, then $w_N \ge ks_n$ if and only if $\frac{1}{k}w_N \ge s_n$, hence by definition, $\frac{1}{k}w_N = v_N$. But now

$$\limsup ks_n = \lim w_N = \lim kv_n = k \lim v_n = k \limsup s_n.$$

b. Do the same for lim inf. Proof 1: replace every "sup" with "inf" above, and "$\ge s_n$" with "$\le s_n$". Proof 2: use the fact that $\liminf s_n = -\limsup(-s_n)$. Then

$$\liminf ks_n = -\limsup k(-s_n) = -k \limsup(-s_n) = k \liminf s_n.$$

c. What happens in (a) and (b) if $k < 0$? We have $\limsup ks_n = k \liminf s_n$ and $\liminf ks_n = k \limsup s_n$. The minimal-work way to do this is to write $k = -c$, where $c > 0$, and apply the first two parts. I'll only do the first.

$$\limsup ks_n = \limsup c(-s_n) = c \limsup(-s_n) = c(-\liminf s_n) = k \liminf s_n.$$

5. – 12.8. Supppose (s_n) and (t_n) are bounded sequences of nonnegative numbers. Show that $\limsup(s_n t_n) \le \limsup s_n \limsup t_n$. Note that $(s_n t_n)$ is also a bounded sequence, so its lim sup is finite. Either approach of 12.4 will work, for reasons of space, I'll do the direct one. Once again, let $a_N = \sup\{s_n : n > N\}$ and $b_N = \sup\{t_n : n > N\}$ and let $d_N = \sup\{s_n t_n : n > N\}$. If $n > N$, then $a_N \ge s_n$ and $b_N \ge t_n$, because they are suprema, and so $a_N b_N \ge s_n t_n$ for all $n > N$. This means that $a_N b_N \ge d_N$, because d_N is supposed to be the smallest number with this property. Taking the limit as $N \to \infty$, we see that $\lim a_N b_N \ge \lim d_N$. But $a_N \to \limsup s_n$ and $b_N \to \limsup t_n$, so $a_N b_N \to (\limsup s_n)(\limsup t_n)$. Since $d_N \to \limsup(s_n t_n)$, we are done.

6. – 13.4. Prove (iii) and (iv) in Discussion 13.7. This is an instance of the homework getting ahead of the lectures. As the book says, a set E in a metric space (S,d) is open if every $x \in E$ is an interior point; that is, if for every $x \in E$, there exists $r > 0$ so that, every y with $d(x,y) < r$ is also contained in E.
(iii) Prove that the union of any collection of open sets is open. Suppose $E = \cup E_j$, and each E_j is open. *I don't know care how many sets there are!* If $x \in E$, then $x \in E_j$ for some j. Since E_j is open, there exists $r > 0$ so that every y with $d(x,y) < r$ is also contained in E_j. But then every such y is also in $\cup E_j = E$, so that E is open.
(iv) Prove that the intersection of a finite collection of open sets is open. Suppose $E = \cap_{j=1}^n E_j$, and each E_j is open. If $x \in E$, then $x \in E_j$ for every j. Since E_j is open, there exists $r_j > 0$ so that every y with $d(x,y) < r_j$ is also contained in E_j. If we let $r = \min\{r_1, \ldots, r_n\} > 0$, then $d(x,y) < r \implies d(x,y) < r_j$. Thus, if $d(x,y) < r$, then y belongs to each E_j and so $y \in \cap E_j = E$, so that E is open.

As I noted on the newsgroup, (iv) can fail for infinite collections of open sets, because $\inf r_j$ might well be zero. The canonical example is $E_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$, $n \in \mathbf{N}$; it's not hard to see that $\cap_{n=1}^\infty E_n = \{0\}$, and so is not an open set.

7. – Find a sequence (s_n) with the property that $\sup s_n = 4$, $\limsup s_n = 3$, $\liminf s_n = 2$ and $\inf s_n = 1$.

One such sequence is $s_1 = 4$, $s_2 = 1$, and $s_n = \frac{1}{2}(5 + (-1)^n)$ for $n \geq 3$; that is, (s_n) takes the values 4,1,2,3,2,3,2,3, etc. Other valid solutions are acceptable of course.

8. – Suppose (s_n) is a bounded, but not convergent, sequence and $s_n > 0$.
a. If $t_n \to t \neq 0$, prove that $(s_n t_n)$ is not convergent.

I think the fastest proof is to observe that, since (s_n) is bounded, but not convergent, we have $\limsup s_n = a$ and $\liminf s_n = b$, where $a > b$. It follows by Theorem 12.1 (and its obvious generalization to lim inf), that $\limsup s_n t_n = ta$ and $\liminf s_n t_n = tb$, and $a > b$ and $t > 0$ imply that $ta > tb$, so $\limsup s_n t_n > \liminf s_n t_n$ and $\lim s_n t_n$ does not exist.
b. Let $s_n = 2 + (-1)^n$ (which satisfies the criteria of this problem.) Find two bounded, but not convergent, sequences (u_n) and (v_n) so that $u_n \geq 1$ and $v_n \geq 1$ and $(s_n u_n)$ is convergent, but $(s_n v_n)$ is not convergent.

The sequence (s_n) takes the values $1, 3, 1, 3, \ldots$. The simplest examples might be to take $u_n = s_{n+1}$, so that (u_n) takes the values $3, 1, 3, 1, \ldots$ and $s_n u_n = 3$ for every n, and to take $v_n = s_n$ so that $s_n v_n$ alternates between 1 and 9. Other correct examples are possible, of course.

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9. – 12.12. Suppose $s_n \geq 0$, and for each n, $\sigma_n = \frac{1}{n}(s_1 + \cdots + s_n)$.

a. Show that $\liminf s_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup s_n$. Well, the middle inequality is immediate. Let me prove that $\limsup \sigma_n \leq \limsup s_n$. If $\limsup s_n = \infty$, there is nothing to prove. Let $\limsup s_n = s$ and fix $\epsilon > 0$, then there exists N so that, for $n > N$, $s_n < s + \epsilon$. Suppose now that $n > N$. What can one say about σ_n for $n > N$? We have

$$\sigma_n = \frac{s_1 + \cdots + s_n}{N} = \frac{s_1 + \cdots + s_N}{n} + \frac{s_{N+1} + \cdots + s_n}{n}.$$

Let $c = s_1 + \cdots + s_N$. There's not a heck of a lot we can say about it. We can say, however, that each of the terms in the numerator of the second fraction is $< s+\epsilon$. Thus,

$$\sigma_n = \frac{c}{n} + \frac{s_{N+1}+\cdots+s_n}{n} < \frac{c}{n} + \frac{(n-N)(s+\epsilon)}{n} = s+\epsilon+\frac{c-N(s+\epsilon)}{n}.$$

This upper bound has a limit of $s+\epsilon$ as $n \to \infty$, because, in the last term, the numerator is a fixed (yet unknowable) constant, and the denominator is going to ∞. It follows that

$$\limsup \sigma_n \le \limsup \left(s+\epsilon+\frac{c-N(s+\epsilon)}{n}\right) = \lim \left(s+\epsilon+\frac{c-N(s+\epsilon)}{n}\right) = s+\epsilon,$$

and since this is true for every ϵ, $\limsup \sigma_n \le s$. The proof for $\liminf$ is nearly identical. In fact, if $\liminf s_n$ is finite, then everything written above can be rewritten in the obvious way, except that there's something to prove if $\liminf s_n = \infty$: that $\liminf \sigma_n = \infty$. The easiest way to do this is the following. If $\liminf s_n = \infty$, then $\lim s_n = \infty$. Given $M > 0$, there exists N so that $n > N$ implies that $s_n > 2M$. Since $s_n > 0$, it follows that for $n > 2N$,

$$\begin{aligned}\sigma_n = \frac{s_1+\cdots+s_n}{N} &= \frac{s_1+\cdots+s_N}{n} + \frac{s_{N+1}+\cdots+s_n}{n} \\ &> 0 + \frac{(n-N)(2M)}{n} = (2-\tfrac{2N}{n})M > M.\end{aligned}$$

Thus $\sigma_n \to \infty$. I will be impressed with anyone who noticed this and tried to prove it.

b. Show that, if $\lim s_n$ exists, then $\lim \sigma_n$ exists, and $\lim \sigma_n = \lim s_n$. This is trivial, given a. If $\lim s_n = s$ exists, then we know that $\liminf s_n = \limsup s_n = s$, and by a. and the squeeze principle, $\liminf \sigma_n = \limsup \sigma_n = s$, so $\lim \sigma_n = s$.

10. – Recall that the *Whitman sequence* (W_n) consists of the ordered blocks of all decimal expansions with n digits, written in increasing order, as n runs from 1 to ∞. That is, the sequence is

$$0, .1, \ldots, .9, 1, 0, .01, \ldots, .99.1, 0, .001, \ldots, .999, 1, 0, .0001, \ldots, .9999, 1, 0, .00001, \ldots$$

We've seen that every $x \in [0,1]$ is a subsequential limit from the Whitman sequence, upon taking the sequence of consecutive decimal approximations. In this problem, your job is to modify the Whitman sequence by creating a sequence (z_n), **which contains $(\mathbf{W}(\mathbf{n}))$ as a subsequence**, and which has the property that its subsequential limits are $\mathbf{R}$. Note: I think Example 3, p.65 is **not** the right approach.

The example I had in mind was to take each of the blocks and make them longer on both ends. Consider, for example, the blocks $\{B_1, B_2, B_3, ...\}$, where

$$\begin{aligned}B_1 &= \{-1, -.9, -.8, \ldots, .8.9, 1.0\}, \\ B_2 &= \{-10, -9.99, -9.98, \ldots, 9.98, 9.99, 10\}, \\ B_3 &= \{-100, -99.999, -99.998, \ldots, 99.998, 99.999, 100\},\end{aligned}$$

etc.

Any real number x has the property that $|x| \le 10^N$ for some N, and we can then be sure that the n-th decimal approximation to x will occur in B_n for $n > N$, and, taking these approximations as a subsequence, we obtain x as a subsequential limit. I neglected to mention that $\pm\infty$ are also subsequential limits.