Math 347 Homework 5 Solutions Due Mon., Oct. 8, 2001

1. – 10.8 (Hint: first prove that $s_n \geq \sigma_n$, and then write σ_n as a linear combination of σ_{n-1} and s_n.) Observe that $n\sigma_n = \sum_{k=1}^{n} s_n$. So, we have

$$n\sigma_n = (n-1)\sigma_{n-1} + s_n \implies n(\sigma_n - \sigma_{n-1}) = s_n - \sigma_{n-1} = s_n - \frac{s_1 + \cdots + s_{n-1}}{n-1}.$$

Since (s_n) is nondecreasing by hypothesis, it follows that $s_n \geq s_j$ for $1 \leq j \leq n-1$. Thus, the final inequality above can be rewritten to complete the proof as

$$s_n - \sigma_{n-1} = s_n - \frac{s_1 + \cdots + s_{n-1}}{n-1} \geq s_n - \frac{s_n + \cdots + s_n}{n-1} = 0.$$

2.,3., – 12.9, 12.13 (ungraded). Answers in back. Ask if you have questions.

4. – 13.10a,b. In a., you want to show that $\{\frac{1}{n} : n \in N\}$ has an empty interior. Well, suppose $x = \frac{1}{n}$. If x were in the interior, then there would exist $r > 0$ so that every real in the interval $(x - r, x + r)$ is the reciprocal of an integer. This is ridiculous. In fact, if $r < \frac{1}{n(n+1)}$, then $x - r > \frac{1}{n+1}$ and $x + r < \frac{1}{n-1}$, so that $\frac{1}{n}$ is the only such point in the interval. In b., the same question is asked for $\mathbf{Q}$. The same proof doesn't work, because there are rational numbers arbitrarily close to each other. Suppose x is in the interior of $\mathbf{Q}$. Then, in particular, $x \in \mathbf{Q}$. As proved in class, or on class notes 7, if any sequence $x_n \to x$, then there would exist N so that, for $n > N$, $x_n \in \mathbf{Q}$. But we've already seen that there exist sequences of irrationals converging to a rational. If $x_n = x + \frac{1}{n}\sqrt{2}$, for example, then $x_n \notin \mathbf{Q}$ for all n, yet $x_n \to x$, a contradiction.

5. – 13.12. (Made optional in class Friday 10/5. I will **not** talk about this during class time.) A set E in a metric space (S, d) is said to be compact if, whenever we have a family of open sets $\mathcal{U}$ so that $E \subset \cup\{U : U \in \mathcal{U}\}$ then there is a finite set $\{U_{n_1}, \ldots, U_{n_k}\}$ so that $E \subset (U_{n_1} \cup \cdots \cup U_{n_k})$. The family $\mathcal{U}$ is said to be a "cover" of E, and a set is compact if every cover has a finite subcover. The reason that $(0, 1)$ is not compact is that

$$(0,1) \subseteq \bigcup_{n=1}^{\infty} \left(\tfrac{1}{n}, 1\right),$$

but any finite union on the right will not contain sufficiently small positive numbers. The Heine-Borel theorem states that if (S, d) is Euclidean R^k space, then a set is compact if and only if it is closed and bounded. We'll get back to this.

In the problem at hand, you are supposed to show that if (S, d) is a metric space and (a) $E \subset F$, where E is closed and F is compact, then E is also compact. Suppose $E \subset \cup\{U : U \in \mathcal{U}\}$. Let's throw in another open set E^c, the complement of E, which is open because E is closed. So what is $\cup\{U : U \in \mathcal{U}\} \cup E^c$? It contains everything in E and everything in E^c, so it contains everything in (S, d). In particular, it contains F and F is compact! Therefore, F is contained in a *finite* subcover; that is, for some $\{n_1, \ldots, n_k\}$,

$$F \subset U_{n_1} \cup \cdots \cup U_{n_k} \qquad \text{or} \qquad F \subset U_{n_1} \cup \cdots \cup U_{n_k} \cup E^c.$$

This is because these are the elements of $\mathcal{U} \cup E^c$. Since E is a subset of F, we have

$$E \subset U_{n_1} \cup \cdots \cup U_{n_k} \qquad \text{or} \qquad E \subset U_{n_1} \cup \cdots \cup U_{n_k} \cup E^c.$$

But E and E^c are disjoint, so we can ignore it, and conclude that E is a subset of a finite number of U_n's. This means that E is compact.

(b) Let $F_1, \ldots, F_n$ be compact and let $F = \cup_{j=1}^{n} F_j$. Suppose $\mathcal{U}$ is a family of open sets which covers F. Then $\mathcal{U}$ covers each F_j, and so $F_j \subset \cup_{i=1}^{n_j} U_{m_{j,i}}$. That is, each F_j has a finite subcover. If we then take the union of these finite subcovers, it is finite and covers all of F. The reason this theorem is limited to a finite number of compact sets is that it is false for infinite unions. Consider, for example, $\cup_{n=1}^{\infty}[\frac{1}{n}, 1] = (0, 1]$.

This problem was probably too hard even if we'd covered it in class.

6. – 14.2a. This first sum, $\sum \frac{n-1}{n^2}$, diverges because $\frac{n-1}{n^2} \geq \frac{n}{2n^2} = \frac{1}{2n}$ for $n \geq 1$ and $\sum \frac{1}{2n}$ diverges.

14.2b. The sum $\sum(-1)^n$ diverges, because $|(-1)^n| = 1$ does not go to zero as $n \to \infty$.

14.2c. The sum $\sum \frac{3n}{n^3} = \sum \frac{3}{n^2}$ converges by the p-test (see p.92, (1)). I suspect this is a typo for $\sum \frac{3^n}{n^3}$, which diverges by the ratio or the root test.

14.2d. The sum $\sum \frac{n^3}{3^n}$ converges by the ratio or root test:

$$\frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}} = \frac{1}{3}\left(\frac{n+1}{n}\right)^3 \to \frac{1}{3}; \qquad \left(\frac{n^3}{3^n}\right)^{1/n} = \frac{(n^{1/n})^3}{3} \to \frac{1}{3}.$$

7. – Let $s_n = 2^{\lfloor n/2 \rfloor}$, so that, for every integer k, $s_{2k} = s_{2k+1} = 2^k$. Evaluate

$$\liminf s_n^{1/n}, \qquad \limsup s_n^{1/n}, \qquad \liminf \frac{s_{n+1}}{s_n}, \qquad \limsup \frac{s_{n+1}}{s_n},$$

and compare with Theorem 12.2.

Notice that $\frac{s_{n+1}}{s_n}$ alternates between 1 and 2, hence $\limsup \frac{s_{n+1}}{s_n} = 2$ and $\liminf \frac{s_{n+1}}{s_n} = 1$. On the other hand

$$s_{2k}^{\frac{1}{2k}} = 2^{\frac{k}{2k}} = 2^{\frac{1}{2}} \qquad \text{and} \qquad s_{2k+1}^{\frac{1}{2k+1}} = 2^{\frac{k}{2k+1}} \to 2^{\frac{1}{2}}.$$

Thus, $s_n^{1/n} \to \sqrt{2}$ and $\liminf s_n^{1/n} = \limsup s_n^{1/n} = \sqrt{2}$. We have $1 \leq \sqrt{2} \leq \sqrt{2} \leq 2$.

8. – Suppose (s_n) and (t_n) are bounded sequences, but not necessarily non-negative and not necessarily convergent, and suppose $\limsup s_n = s$ and $\limsup t_n = t$. Is it a correct theorem that $\limsup(s_n + t_n) = s + t$? Is it a correct theorem that $\limsup(s_n t_n) = st$? This

problem requires either a proof similar to that on the last homework, or a counterexample or both.

Heh-heh. This is too easy. If (s_n) is the sequence that alternates $0, 1, 0, 1, \ldots$ and (t_n) is the sequence that alternates $1, 0, 1, 0, \ldots$, then $s = t = 1$, but $s_n + t_n$ is the constant sequence of 1's and $s_n t_n$ is the constant sequence of 0's. Thus, neither is a correct theorem. I *intended* to ask this question with $\leq$ instead of $=$. See Homework 6, #8.

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9. – Suppose (s_n) is a sequence with the property that for every k,

$$|s_{2k+1} - s_{2k}| < \frac{1}{k} \qquad \text{and} \qquad |s_{2k+2} - s_{2k+1}| < \frac{1}{2^k}.$$

Must it be true that (s_n) is a Cauchy sequence? (Proof, or counterexample.)

No. The example I had in mind is

$$s_{2k+1} = s_{2k+2} = \sum_{j=1}^{k+1} \frac{1}{j}.$$

In this case, $|s_{2k+1} - s_{2k}| = \frac{1}{k+1} < \frac{1}{k}$ and $|s_{2k+2} - s_{2k+1}| = 0 < \frac{1}{2^k}$. In this case $s_n \to \infty$ is divergent, and so the sequence could not be Cauchy. Other examples are possible of course.

10. – Using Theorem 12.2, or any correct method, compute

$$\lim \left(\frac{(2n)!}{(n!)^2} \right)^{1/n}, \qquad \lim \left(\frac{(5n)!}{((2n)!)^2} \right)^{1/n}, \qquad \lim \left(\frac{(mn)!}{(n!)^m} \right)^{1/n} \quad \text{for fixed } m.$$

Hint: $(2(n+1))! = (2n+2)(2n+1)(2n)!$, etc.

I've corrected the second one, as I noted in the newsgroup and the webpage. In each case, Theorem 12.2 and Corollary 12.3 suggest a comparison of consecutive ratios. If they converge to L, then the n-th roots will also converge to L. In the first case, we have

$$\frac{\frac{(2n+2)!}{((n+1)!)^2}}{\frac{(2n)!}{(n!)^2}} = \frac{(2n+2)!}{(2n)!} \cdot \frac{(n!)^2}{((n+1)!)^2} = \frac{(2n+1)(2n+2)}{(n+1)^2} = \frac{4n^2+6n+2}{n^2+2n+1} \to 4.$$

In the second case,

$$\frac{\frac{(5n+5)!}{((2n+2)!)^2}}{\frac{(5n)!}{((2n)!)^2}} = \frac{(5n+5)!}{(5n)!} \cdot \frac{((2n)!)^2}{((2n+2)!)^2} = \frac{(5n+1)(5n+2)(5n+3)(5n+4)(5n+5)}{(2n+1)^2(2n+2)^2} \to \infty,$$

since the numerator has a higher degree than the denominator. Since $\liminf \frac{s_{n+1}}{s_n} = \infty$, it follows from 12.2 that $s_n^{1/n} \to \infty$.

The third case is essentially the same as the first. We have

$$\frac{\frac{(mn+m)!}{((n+1)!)^m}}{\frac{(mn)!}{(n!)^m}} = \frac{(mn+m)!}{(mn)!} \cdot \frac{(n!)^m}{((n+1)!)^m} = \frac{(mn+1)\ldots(mn+m)}{(n+1)^m} = \frac{m^m n^m + \ldots}{n^m + \ldots} \to m^m.$$