Math 347 Homework 6 Solutions Due Mon., Oct. 15, 2001

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1. and 4. – 14.3 – a,b,c,d and 14.7 (ungraded) See answers in back. Ask if unclear.

2a. Observe that the dominant term is n and $n + (-1)^n \geq n - 1$, hence for $n \geq 2$, $\frac{1}{(n+(-1)^n)^2} \leq \frac{1}{(n-1)^2}$. Since $\sum \frac{1}{(n-1)^2}$ is convergent (by an index-shift and the p-test with $p = 2$, or by $(n-1)^2 \geq n^2/4$ or by the theorem in class and on Bonus Notes 9, considering $\frac{p(n)}{q(n)}$ with $p(x) = 1$ and $q(x) = (x-1)^2$), this series is convergent.

b. There are two ways to do this. One is to let $a_n = \sqrt{n+1} - \sqrt{n}$ and then observe that the partial sums telescope to $s_n = \sqrt{n+1} - 1$. Since $\sqrt{n+1} \to \infty$, the series diverges. The other way to do this is to exploit the algebraic identity:

$$\sqrt{n+1} - \sqrt{n} = \left(\sqrt{n+1} - \sqrt{n}\right) \cdot \left(\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}\right) = \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

This suggests divergence, and in fact, if you use the inequality $\sqrt{n+1} - \sqrt{n} > \frac{1}{2\sqrt{n+1}}$, then the comparison test and the p-test with $p = \frac{1}{2}$ shows divergence.

c. Using the hint of the root test, we have $\left(\frac{n!}{n^n}\right)^{1/n} = \frac{(n!)^{1/n}}{n} \to \frac{1}{e} < 1$ as shown in class, so the series converges. If you forgot that, you can still use the ratio test, and obtain

$$\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \frac{(n+1)!}{(n+1) \cdot n!} \cdot \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n \to \frac{1}{e}.$$

(This computation was how we got the first limit anyway.)

3. – 14.6a. Suppose $\sum |a_n|$ converges and (b_n) is bounded, say $|b_n| \leq M$ for all n. (a) Show that $\sum a_n b_n$ converges. Following the hint, we'll prove that it's Cauchy. Indeed, since $\sum |a_n|$ converges, it is Cauchy, and for every $\epsilon > 0$ there exists N so that for $m, n > N$, $\sum_{k=m+1}^{n} |a_k| < \epsilon$. So, given $\epsilon > 0$, there exists N' so that $m, n > N'$, $\sum_{k=m+1}^{n} |a_k| < \epsilon/M$. We do this so we can make the following estimate, showing that $\sum a_n b_n$ is Cauchy:

$$\left|\sum_{k=m+1}^{n} a_k b_k\right| \leq \sum_{k=m+1}^{n} |a_k||b_k| \leq \left(\sum_{k=m+1}^{n} M \cdot |a_k|\right) < M \cdot \frac{\epsilon}{M} = \epsilon.$$

The point in (b) is that $a_n = (\pm 1)|a_n|$, depending on the sign of a_n, and a sequence where $b_n = \pm 1$ is bounded.

5. – 14.8. Use the hint! Suppose x and y are non-negative numbers. We have

$$0 \leq (\sqrt{x} - \sqrt{y})^2 = x - 2\sqrt{xy} + y \implies \sqrt{xy} \leq \frac{x+y}{2} \leq x + y.$$

Let (s_n) and (t_n) denote the partial sums of (a_n) and (b_n) respectively. In view of the hint, and because $a_n, b_n \geq 0$, we have

$$\sum_{k=1}^{n} \sqrt{a_k b_k} \leq \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = s_n + t_n.$$

Since (s_n) and (t_n) are bounded non-decreasing sequences, so is the sequence of partial sums for the given series, and hence it is convergent.

6. – 14.12. Given a sequence (a_n) so that $\liminf |a_n| = 0$, we wish to find a subsequence a_{n_k} so that $\sum_{k=1}^{\infty} a_{n_k}$ is convergent. In a problem like this, since the information is about $|a_n|$, you want to prove that the sum of the subsequence is absolutely convergent, and the easiest thing is to find a subsequence so that $a_{n_k} \leq \frac{1}{2^k}$. How can this be done? We know that $\liminf |a_n| = 0$; thus if $u_N = \inf\{|a_n| : n > N\}$, then $u_N \geq 0$ for all n (since 0 is a lower bound) and (u_n) is non-decreasing, hence $u_N = 0$ for all N. Thus, for all k and all N, $2^{-k} > u_N$, thus there exists $n > N$ so that $2^{-k} > |a_n|$. So here's what we do. We pick n_1 so that $|a_{n_1}| \leq 1/2$. Then we pick $n_2 > n_1$ so that $|a_{n_2}| \leq 1/4$, etc. This gives the desired subsequence so that $|a_{n_k}| \leq 2^{-k}$, and $\sum a_{n_k}$ is absolutely convergent and hence is convergent.

7. – 16.4 - c,f. We have, for the first,

$$.\overline{02} = \frac{0}{10^1} + \frac{2}{10^2} + \frac{0}{10^3} + \frac{2}{10^4} + \cdots = 2\sum_{k=1}^{\infty} \frac{1}{10^{2k}} = \frac{2}{1 - \frac{1}{100}} = \frac{2}{99}.$$

For the second, a little more care is required,

$$\begin{aligned} .1\overline{492} &= \frac{1}{10^1} + \frac{4}{10^2} + \frac{9}{10^3} + \frac{2}{10^4} + \frac{4}{10^5} + \frac{9}{10^6} + \frac{2}{10^7} + \cdots = \frac{1}{10^1} + \frac{492}{10^4} + \frac{492}{10^7} + \ldots \\ &= \frac{1}{10^1} + \frac{492}{10^4} \cdot \left(\sum_{k=0}^{\infty} \frac{1}{10^{3k}} \right) = \frac{1}{10^1} + \frac{492}{10^4} \cdot \frac{1000}{999} = \frac{1}{10} + \frac{492}{9990} = \frac{1491}{9990}. \end{aligned}$$

8. – (The correction.) Suppose (s_n) and (t_n) are bounded sequences, but not necessarily non-negative and not necessarily convergent, and suppose $\limsup s_n = s$ and $\limsup t_n = t$. Is it a correct theorem that $\limsup(s_n + t_n) \leq s + t$? Is it a correct theorem that $\limsup(s_n t_n) \leq st$? This problem requires either a proof similar to that on the last homework, or a counterexample or both.

I think I've made much too big a deal out of this. The theorem is correct for the sum and false for the product. Following the old proof, given $\epsilon > 0$, $\limsup s_n = s$ implies that there exists N_1 so that $\sup\{s_n : n > N_1\} < s + \epsilon/2$, hence for each $n > N_1$, $s_n < s + \epsilon/2$. Similarly, there is N_2 so that for $n > N_2$, $t_n < t + \epsilon/2$, and thus, if $N = \max\{N_1, N_2\}$, then for $n > N$, $s_n + t_n < s + t + \epsilon$. It follows that $\limsup(s_n + t_n) < s + t + \epsilon$. This is true for every $\epsilon > 0$ and so $\limsup(s_n + t_n) < s + t$. The point here is that non-negativity isn't a factor when you're adding.

For multiplication, it's a different story. The simplest counterexample has $(s_n) = (t_n)$ being the sequence alternating between 0 and -1. Then $s = t = 0$, but $(s_n t_n)$ is the sequence alternating between 0 and $(-1)^2 = 1$, so $\limsup(s_n t_n) = 1$, which is not ≤ 0.

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9. – Observe that for every $n \in N$, there exists $r \geq 0$ so that $2^r \leq n \leq 2^{r+1} - 1$. (In fact, $r = \lfloor \log_2 n \rfloor$.) Define a sequence (s_n) as follows: if $2^r \leq n \leq 2^{r+1} - 1$ and r is even, then

$s_n = 1$; if r is odd, then $s_n = 0$. Thus, for $n = 1, 2, 3, 4, 5, 6, 7$, we have $r = 0, 1, 1, 2, 2, 2, 2$ and $s_n = 1, 0, 0, 1, 1, 1, 1$. As before, let $\sigma_n = \frac{1}{n}(s_1 + \cdots + s_n)$. Compute σ_n when $n = 2^r + j$, $0 \leq j \leq 2^r - 1$ and determine $\liminf \sigma_n$ and $\limsup \sigma_n$. (Hint: first compute $\sigma_{2^{2k}}$ and $\sigma_{2^{2k+1}}$.)

Perhaps a better hint would have been to compute $\sigma_{2^r - 1}$. Let $x_n = s_1 + \cdots + s_n$. Then $x_1 = 1, x_3 = 1$, $x_7 = 5$ from the information given above, and $x_{15} = 5$ as well, because for $8 \leq n \leq 15$, we have $r = 3$ and $s_n = 0$. But for $16 \leq n \leq 31$, we have $r = 4$ and $s_n = 1$, so that $x_{31} = 5 + 16 = 21$, and for $32 \leq n \leq 63$, we have $r = 5$ and $s_n = 0$, so $x_{63} = 21$ as well. I hope you can see the pattern: $x_n = x_{n-1} + 1$ if and only if r is even, so there are blocks in which x_n is incrementing by 1 and blocks in which it is constant.

$$x_1 = x_3 = 1; x_4 = 2, \ldots, x_6 = 4 x_7 = x_{15} = 5; x_{16} = 6, \ldots, x_{30} = 20, x_{31} = x_{63} = 21; \ldots.$$

The relevance of $1, 5, 21$ is that they are $1, 1 + 4, 1 + 4 + 16$. In fact, it's easy to see that

$$x_{2^{2k+1}-1} = x_{2^{2k+2}-1} = 1 + 4 + \cdots + 4^k = \frac{4^{k+1} - 1}{4 - 1} \approx \frac{2}{3} \cdot 2^{2k+1}.$$

It follows from this that $\sigma_{2^{2k+1}-1} \to \frac{2}{3}$ and $\sigma_{2^{2k+2}-1} \to \frac{1}{3}$.

What about the other σ_n's? Well, notice that $s_n = 0$ or 1 and so $0 \leq \sigma_n \leq 1$. Furthermore, if $s_n = 1$, then $\sigma_n \leq \sigma_{n+1}$ and if $s_n = 0$, then $\sigma_n \geq \sigma_{n+1}$. We see then that σ_n is increasing from $n = 2^{2k} - 1$ to $n = 2^{2k+1} - 1$ and decreasing from $n = 2^{2k+1}$ to $n = 2^{2k+2} - 1$, so the values found above in fact show that $\limsup \sigma_n = \frac{2}{3}$ and $\liminf \sigma_n = \frac{1}{3}$.

10. – Construct a sequence (s_n) so that

$$\liminf \frac{s_{n+1}}{s_n} < \liminf s_n^{1/n} < \limsup s_n^{1/n} < \limsup \frac{s_{n+1}}{s_n}$$

Suggestion: Emulate the example of Homework 5, #7, but make different rules for $\frac{s_{n+1}}{s_n}$ depending on whether $2^{2k} \leq n < 2^{2k+1}$ or $2^{2k+1} \leq n \leq 2^{2k+2}$. Suppose we follow the pattern of problem 9, and say that $\frac{s_{n+1}}{s_n} = 2$ if $2^{2k} \leq n < 2^{2k+1}$ and $\frac{s_{n+1}}{s_n} = 1$ if $2^{2k+1} \leq n < 2^{2k+2}$. Then $\frac{s_{n+1}}{s_n}$ takes the values 1 or 2 infinitely often, hence their liminf and limsup are 1 and 2 respectively.

What is s_n? Nothing less that 2^{x_n} from the last problem! What is $s_n^{1/n}$? Nothing less that 2^{σ_n} from the last problem! Therefore, we have

$$\liminf \frac{s_{n+1}}{s_n} = 1 < \liminf s_n^{1/n} = 2^{1/3} < \limsup s_n^{1/n} = 2^{2/3} < \limsup \frac{s_{n+1}}{s_n} = 2.$$

Naturally, other examples are possible and will be carefully considered.