Math 347 **Homework 7** **Due Wed., Oct. 24, 2001**

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1. – 17.1 (ungraded); 3. 17.9 (ungraded). Answers in back. Ask if you have a question.

2. – 17.2. Here, $f(x) = 4$ if $x \geq 0$ and $f(x) = 0$ if $x < 0$ and $g(x) = x^2$ for all real x, and $\mathrm{dom}(f) = \mathrm{dom}(g) = \mathbf{R}$.
(a) Then $(f+g)(x) = 4 + x^2$ if $x \geq 0$ and $(f+g)(x) = x^2$ if $x < 0$; $(fg)(x) = 4x^2$ if $x \geq 0$ and $(fg)(x) = 0$ if $x < 0$; $(f \circ g)(x) = f(g(x)) = f(x^2) = 4$ (for all x) and $(g \circ f)(x) = g(f(x)) = (f(x))^2 = 16$ if $x \geq 0$ and $(g \circ f)(x) = 0$ if $x < 0$. In each of these cases, the domain is $\mathbf{R}$.
(b) Recall that polynomials are continuous and that if f jumps at $x = a$, then it is not continuous at a, because, if $s_n \to a$ and $s_n < a$ and $t_n \to a$, but $t_n > a$, then $(f(s_n))$ and $(f(t_n))$ will converge to different values. On this basis, g, and $f \circ g$ are polynomials, and so are continuous, but f, $f+g$ and $g \circ f$ jump at 0, and so are not continuous there. Note that fg is not a polynomial, but it doesn't have a jump at $x = 0$, so it's continuous. (See Homework 8, #8.)

4. – 18.6. Assuming that $\cos x$ is continuous, let $f(x) = \cos x - x$, which is also continuous. We have $f(0) = \cos 0 - 0 = 1$ and $f(\frac{\pi}{2}) = \cos(\frac{\pi}{2}) - \frac{\pi}{2} = -\frac{\pi}{2}$. Thus, $f(0) > 0 > f(\frac{\pi}{2})$. By the Intermediate Value Theorem, it follows that there exists $x_0 \in [0, \frac{\pi}{2}]$ so that $f(x_0) = \cos x_0 - x_0 = 0$.

5. – 18.10. So, suppose $f(0) = f(2)$ and f is continuous on $[0, 2]$. Let $g(x) = f(x+1) - f(x)$. Then g is continuous on $[0, 1]$ and $g(0) + g(1) = f(2) - f(0) = 0$. Thus, either $g(0) = g(1) = 0$, which means that $f(0) = f(1) = f(2)$, or $\{g(0), g(1)\}$ consists of one positive and one negative number. No matter which, the IVT implies that there exists $x \in [0, 1]$ so that $g(x) = 0$; that is, $f(x) = f(x+1)$.

6. – 19.2a. Observe that $|f(x) - f(y)| = |(3x+11) - (3y+11)| = 3|x - y|$. So, given $\epsilon > 0$, we see that

$$|x - y| < \frac{\epsilon}{3} \implies |f(x) - f(y)| = 3|x - y| < \epsilon.$$

7. – 19.2b. With $f(x) = x^2$ on [0,3], we have

$$|f(x) - f(y)| = |x^2 - y^2| = |x - y| \cdot |x + y| \leq |x - y| \cdot (3 + 3) = 6|x - y|.$$

So, given $\epsilon > 0$, we see that

$$|x - y| < \frac{\epsilon}{6} \implies |f(x) - f(y)| \leq 6|x - y| < 6 \cdot \frac{\epsilon}{6} = \epsilon.$$

8. – 19.6. My hint is for a really fast way to do a. and b. at once. I'll start with an acceptable solution to the problem that ignores the hint.

a. If $f(x) = \sqrt{x}$, then $f'(x) = \frac{1}{2\sqrt{x}}$ is unbounded on $(0,1]$. This is obvious, but it's also easily proved: if $x_n = n^{-2} \in (0,1]$, then $f'(x_n) = n/2$, which is unbounded as $n \to \infty$. On the other hand, f is the inverse of the (continuous, monotone) polynomial x^2, so it is continuous by Theorem 18.4. Theorem 19.2 says that if f is continuous on a closed interval $[a,b]$, then f is uniformly continuous on $[a,b]$, so f is uniformly continuous on $[0,1]$.
b. To show that f is uniformly continuous on $[1,\infty)$, observe that

$$\sqrt{x} - \sqrt{y} = \frac{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})}{\sqrt{x} + \sqrt{y}} = \frac{x - y}{\sqrt{x} + \sqrt{y}}.$$

If $x, y \geq 1$, then $\sqrt{x}, \sqrt{y} \geq 1$, so the denominator above is at least 2 and

$$\left|\sqrt{x} - \sqrt{y}\right| \leq \frac{|x-y|}{2}.$$

Thus, if ϵ is given, then $|x - y| < 2\epsilon \implies |f(x) - f(y)| < \epsilon$, and so f is uniformly continuous.

Here is my proof that f is uniformly continuous on $[0,\infty)$, and this subsumes the results above.
Lemma: Fix $t > 0$. Let $\Phi_t(x) = \sqrt{x+t} - \sqrt{x}$ for $x \geq 0$. Then Φ_t is strictly decreasing.
Proof: Observe that

$$\Phi_t(x) = \sqrt{x+t} - \sqrt{x} = \frac{1}{\sqrt{x+t} + \sqrt{x}}.$$

Since $x < y$ clearly implies $\sqrt{x+t} + \sqrt{x} < \sqrt{y+t} + \sqrt{y}$, it follows that $\Phi_t(x) > \Phi_t(y)$. Taking $y = 0$, we see that $\sqrt{x+t} - \sqrt{x} < \sqrt{t} - -\sqrt{0} = \sqrt{t}$. Putting $t = x - y$, we find the key inequality

$$\sqrt{y} - \sqrt{x} < \sqrt{y - x}.$$

(This can also be shown directly, by squaring both sides and transposing.)

Now, suppose $x, y \geq 0$ and $|x - y| < \delta$. Without loss of generality, suppose that $x \geq y$, so $\sqrt{x} \geq \sqrt{y}$. Then $\sqrt{x} - \sqrt{y} < \sqrt{x-y} < \sqrt{\delta}$. Thus, if $\epsilon > 0$ is given, then

$$|x - y| < \epsilon^2 \implies |\sqrt{x} - \sqrt{y}| < \epsilon.$$

The proof applies to any function f which is a monotone increasing continuous function on $[0,\infty)$ satisfying $f(0) = 0$ and so that $x \geq y$ implies $f(x) - f(y) \leq dvif(x-y)$.

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9. – 17.14. This is actually a famous function f so that, if $x \in \mathbf{Q}$ is given in lowest terms as $x = \frac{p}{q}$, then $f(x) = \frac{1}{q}$, and if x is irrational, then $f(x) = 0$. We have to show that f is continuous at each irrational x and not continuous at each rational x.

For the first, suppose x is irrational and suppose $\epsilon < 1$ is given. Choose $N < \frac{1}{\epsilon}$. The interval $(x-1, x+1)$ contains only finitely many rational numbers with denominator $\leq N$. (In fact, each denominator r occurs less than $2r$ times.) None of these rational numbers equals x, so there exists $\delta > 0$ so that if $|y-x| < \delta$ and y is rational, then the denominator of y is $\geq N$. This means that $f(y) \leq \frac{1}{N} < \epsilon$ if y is rational, and of course $f(y) = 0$ if y is irrational. Therefore, $|x-y| < \delta$ implies that $|f(x) - f(y)| < \epsilon$, and this means that f is continuous at x.

If x is rational, then $x_n = x + \frac{1}{n}\sqrt{2}$ is irrational and $\lim x_n = x$. But $f(x_n) = 0$ for all n, so $\lim f(x_n) = 0 \neq f(x)$, and so f is not continuous at x.

One purpose of this mind-bogglingly non-intuitive example is to show how mind-bogglingly non-intuitive real analysis can be.

10a. – This is known as the "Cauchy Condensation Theorem". Suppose (a_n) is a decreasing sequence of positive real numbers, and let $b_n = a_{2^n}$. Prove that $\sum a_n$ is convergent if and only if $\sum 2^n b_n$ is convergent. (Hint: the proof of the p-test on Bonus Notes 8.)

Let $s_m = \sum_{n=1}^m a_n$. Then (s_m) is an increasing sequence, so it's either bounded above or diverges to ∞. Following the hint, observe that, if $2^r \leq n < 2^{r+1}$, then $a_{2^r} \geq a_n > a_{2^{r+1}}$; that is, $b_r \geq a_n > b_{r+1}$. Thus, we have for this block of $2^{r+1} - 2^r = 2^r$ terms

$$2^r b_r \geq a_{2^r} + \cdots + a_{2^{r+1}-1} > 2^r b_{r+1},$$

and so, summing from $r = 0$ to $N-1$,

$$\sum_{r=0}^{N-1} 2^r b_r \geq \sum_{r=0}^{N-1} (a_{2^r} + \cdots + a_{2^{r+1}-1}) = \sum_{n=1}^{2^N-1} a_n \geq \sum_{r=0}^{N-1} 2^r b_{r+1} = \tfrac{1}{2}\sum_{r=1}^{N} 2^r b_r.$$

If $\sum 2^n b_n$ is convergent, then the left hand side is bounded above, and so $\sum a_n$ is convergent. If $\sum 2^n b_n$ is divergent, then the right hand side is unbounded and so $\sum a_n$ is divergent. With $a_n = n^{-p}$, $2^n b_n = 2^n (2^n)^{-p} = (2^{1-p})^n$, and the ratio or root test shows convergence if $p > 1$ and divergence if $p \leq 1$.

10b. Use (a) to determine the values of p for which

$$\sum \frac{1}{n(\log n)^p}$$

converges. We want to use (a), so we need to check that this is decreasing, or, equivalently, that $\phi(x) = x(\log x)^p$ is increasing. Notice that if $p < 0$, then the series diverges with comparison to $\sum \frac{1}{n}$, so we can assume that $p \geq 0$, and then $\phi'(x) = (\log x)^p + px(\log x)^{p-1}(1/x) \geq 0$, at least for $x > e$, which is good enough. Using the preceding, we have

$$2^n a_{2^n} = \frac{2^n}{2^n(\log 2^n)^p} = \frac{1}{(n\log 2)^p} = \frac{1}{(\log 2)^p} \cdot \frac{1}{n^p}.$$

By the p-test, this is convergent if $p > 1$ and divergent if $p \leq 1$. This problem can also be done by the integral test; noting that

$$\int \frac{dx}{x(\log x)^p} = \begin{cases} \frac{1}{1-p} \cdot \frac{1}{(\log x)^{p-1}} + C, & \text{if } p \neq 1, \\ \log\log x + C, & \text{if } p = 1. \end{cases}$$