Math 347 **Homework 8 Solutions** **Due Wed., Oct. 31, 2001**

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1. – 17.10 abc. I will take a sequence of points $x_n \to x_0$ with the property that $f(x_n)$ does not converge to $f(x_0)$ in each case. It is utterly irrelevant that there may be other sequences $y_n \to x_0$ for which $f(y_n) \to f(x_0)$! In (a), if $x_n = \frac{1}{n}$, then clearly $x_n \to 0$ and $f(x_n) = 1$ for all n, but $f(0) = 0$. In (b), taking $x_n = \frac{1}{(2n+\frac{1}{2})\pi}$, we again have $x_n \to 0$, because it is the reciprocal of a sequence going to ∞, and $f(x_n) = \sin(2n + \frac{1}{2})\pi = 1$ for all n, so that $f(x_n) \to 1$, but $f(0) = 0$ again. In (c), we can use the same sequence as in (a), with the same values and the same consequence.

2. – 19.4a. Suppose f is uniformly continuous on a bounded set S; we want to show that f is bounded. Suppose f is not bounded, then for every n, there exists $x_n \in S$ so that $|f(x_n)| > n$. Since S is bounded, (x_n) is a bounded sequence of real numbers, so there exists a subsequence (x_{n_k}) which converges (in $\mathbf{R}$!), **there's no guarantee that the limit is in S, which is not assumed to be a closed set!** Since it is a convergent real sequence, it is Cauchy, and by Theorem 14.4, it follows that $(f(x_n))$ is also Cauchy. But Cauchy sequences are bounded and $(f(x_{n_k}))$ evidently is not bounded, a contradiction. (b) Since $(0, 1)$ is a bounded set and $\frac{1}{x^2}$ is unbounded on this set, it cannot be uniformly continuous.

Note that step functions are examples of bounded functions that aren't continuous, and so aren't uniformly continuous, so the implication only goes in one direction.

3.,6. – 19.9, 20.11. Ask if you have questions.

4. – 19.10. First, g is obviously continuous at x_0 for any $x_0 \neq 0$, and if $x_n \to 0$, then $|g(x_n)| \leq x_n^2$, so $g(x_n) \to 0 = g(0)$. To show that g is uniformly continuous on a bounded set S, it suffices to show that g is differentiable and $|g'|$ is bounded. A calculation shows that

$$g'(x) = 2x\sin(\tfrac{1}{x}) + x^2 \cos \tfrac{1}{x}(-\tfrac{1}{x^2}) = 2x\sin(\tfrac{1}{x}) - \cos(\tfrac{1}{x}),$$

which is bounded by $2|x| + 1$. What happens for large x? This is a little tricky, but as shown in class,

$$\lim_{x\to\pm\infty} 2x\sin(\tfrac{1}{x}) - \cos(\tfrac{1}{x}) = \lim_{x\to 0^{\pm}} 2 \cdot \frac{\sin x}{x} - \cos x = 2 - 1 = 1.$$

This means that, taking $\epsilon = .1$ as a semi-random choice, there exist $M_1 > 0$ and $-M_2 < 0$ so that $x > M_1 \implies g'(x) < 1.1$ and $x < -M_2 \implies g'(x) < 1.1$. In other words, for all $x \in \mathbf{R}$, $|g'(x)| \leq \max\{1.1, 2M_1 + 1, 2M_2 + 1\}$. That is, $|g'|$ is bounded on $\mathbf{R}$. It follows by Theorem 19.6, g is uniformly continuous on $\mathbf{R}$. This was a hard problem.

5. – 20.4 and 20.8. A sketch is Figure 19.3. Let $f(x) = x\sin(\frac{1}{x})$, so that f is not defined at $x = 0$. Then we would intuit that

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x) = 0.$$

These can be proved as follows: we know that f is defined on $(0, \epsilon)$ and on $(-\epsilon, 0)$ for any $\epsilon > 0$ and we also know that $|f(x)| \le |x|$, so if $x_n \to 0$, then $|f(x_n)| \to 0$, hence $f(x_n) \to 0$. It doesn't matter if $x_n > 0$ or $x_n < 0$ in this case.

It's less intuitive what happens as $|x| \to \infty$, but as noted in class,

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to 0^{\pm}} f(\tfrac{1}{x}).$$

In this case, $f(\frac{1}{x}) = \frac{\sin x}{x}$, and as noted in Example 19.9, this limit is 1.

7. – 20.12. (a) Sketch at the end. (b),(c) Write $x = 2 + t$, so that

$$f(x) = \frac{1}{(x-1)(x-2)^2} \implies f(2+t) = \frac{1}{(1+t)t^2}.$$

If t is small and positive, or small and negative, then $f(2+t)$ will be large and positive. In fact, if $\delta < 1/2$, then $|x - 1| < \delta$ (that is, $|t| < \delta$) implies $1 + t < 3/2$, so

$$f(x) = f(2+t) > \frac{1}{(\frac{3}{2})t^2} = \frac{2/3}{t^2}.$$

It's easy to give a δ/ϵ argument (if needed) to show that

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = \lim_{x \to 2} f(x) = \infty.$$

But if $x = 1 + u$ is close to 1, then $f(1+u) = \frac{1}{u(-1+u)^2}$, and if u is small and positive then $f(1+u)$ is large and positive, but if u is small and negative then $f(1+u)$ is large and negative. In this case,

$$\lim_{x \to 1^+} f(x) = \infty, \qquad \lim_{x \to 1^-} f(x) = -\infty,$$

so $\lim_{x \to 1} f(x)$ does not exist.

8. – Suppose $p(x)$ and $q(x)$ are polynomials and suppose $p(a) = q(a)$ for some $a \in \mathbf{R}$. Define

$$f(x) = \begin{cases} p(x), & \text{if } x \le a, \\ q(x), & \text{if } x > a. \end{cases}$$

Prove carefully (that is, with an ϵ-δ argument), that f is continuous at $x = a$. You may assume without proof that p and q are continuous on $\mathbf{R}$.

Let $p(a) = q(a) = L$. Both p and q are continuous at $x = a$, hence, given $\epsilon > 0$, there exists $\delta_1, \delta_2 > 0$ so that

$$|x - a| < \delta_1 \implies |p(x) - L| < \epsilon, \qquad |x - a| < \delta_2 \implies |q(x) - L| < \epsilon.$$

Let $\delta = \min\{\delta_1, \delta_2\}$ and suppose $|x - a| < \delta$. If $x \in (a - \delta, a]$, then $f(x) = p(x)$, so that $|f(x) - L| < \epsilon$. Similarly, if $x \in (a, a + \delta)$, then $f(x) = q(x)$, so that again, $|f(x) - L| < \epsilon$.

This actually shows that any two continuous functions which agree at $x = a$ can be "pasted together" to make a continuous function.

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9. – 20.18. (You may use L'Hopital's Rule *informally*, to determine the limit, but you have to justify your claims rigorously.) It's actually better to rationalize the numerator. We have, for $x \neq 0$,

$$\frac{\sqrt{1+3x^2}-1}{x^2} = \frac{\sqrt{1+3x^2}-1}{x^2} \cdot \frac{\sqrt{1+3x^2}+1}{\sqrt{1+3x^2}+1} = \frac{3x^2}{(\sqrt{1+3x^2}+1)x^2} = \frac{3}{\sqrt{1+3x^2}+1}.$$

Since $\lim_{x\to 0} 3 = 3$ and $\lim_{x\to 0} \sqrt{1+3x^2}+1 = 2$, we have by Theorem 20.4(iii) that the desired limit is $3/2$.

If you want to do this by ϵ/δ, I think that rationalization is still the way to begin. Observe that the denominator is always ≥ 2, so $f(x) \leq \frac{3}{2}$ for all x. Thus, to find δ so that $|x| < \delta \implies |f(x) - \frac{3}{2}| < \epsilon$, it suffices to show that $f(x) > \frac{3}{2} - \epsilon$. A little algebra shows that

$$\frac{3}{\sqrt{1+3x^2}+1} > \frac{3}{2} - \epsilon \iff |x| < \frac{\sqrt{8\epsilon}}{3-2\epsilon}.$$

There is no need for you to have done this calculation if you didn't want to.

10. Let f be the function in 19.9; that is, $f(x) = x\sin(\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0$. We know from general principles that f is uniformly continuous on $[0,1]$. In this problem, you will prove one instance of it directly.

Find, with proof, $\delta > 0$ with the property that for $0 \leq x, y, \leq 1$,

(*) $$|x - y| < \delta \implies |f(x) - f(y)| < \frac{1}{347}.$$

You may use the Mean Value Theorem, and you do *not* have to find the "best" (that is, the largest) δ for which (*) holds. On the other hand, you *do* have to prove it!

I will prove, more generally, that if $\epsilon < \frac{2}{3}$, then

$$|x - y| < \frac{\epsilon^2}{4} \implies |f(x) - f(y)| < \epsilon.$$

Suppose $\epsilon > 0$ is given. Consider two cases: first suppose that $0 \leq x, y < \frac{\epsilon}{2}$. Then

$$|f(x) - f(y)| \leq |f(x)| + |f(y)| \leq x + y < \epsilon.$$

Now suppose $\frac{\epsilon}{3} \leq x, y \leq 1$. We have

$$f'(x) = \sin(\tfrac{1}{x}) + x\cos(\tfrac{1}{x})(-\tfrac{1}{x^2}) \implies |f'(x)| \leq 1 + \frac{1}{x} \leq 1 + \frac{3}{\epsilon}.$$

Then by Theorem 19.6, applied to the closed interval $[\frac{\epsilon}{3}, 1]$, if $|x-y| < \delta$, then

$$|f(x) - f(y)| < \delta \cdot \left(1 + \frac{3}{\epsilon}\right) = \frac{\epsilon^2}{4} \cdot \left(1 + \frac{3}{\epsilon}\right) = \frac{\epsilon(3+\epsilon)}{4} < \epsilon.$$

Ah, but you say, this has only taken care of two cases: $x, y \in [0, \frac{\epsilon}{2})$ and $x, y \in [\frac{\epsilon}{3}, 1]$! But suppose without loss of generality that $x > y$ and $x \geq \frac{\epsilon}{2}$. Then

$$y \geq x - \delta \geq \frac{\epsilon}{2} - \delta = \frac{\epsilon}{2} - \frac{\epsilon^2}{4} = \frac{\epsilon}{3} + \frac{(2-3\epsilon)\epsilon}{12},$$

and since $\epsilon < \frac{2}{3}$, it follows that $x \geq y \geq \frac{\epsilon}{3}$.

This argument, applied to $\epsilon = \frac{1}{347}$ shows that we can take

$$\delta = \frac{1}{4 \cdot 347^2} = \frac{1}{481636}.$$

What's the best that δ could be? Here's a crude estimate. Recall that $\sin(2n + \frac{1}{2})\pi = 1$ and $\sin(2n+\frac{3}{2})\pi = -1$. Let $x_n = ((2n+\frac{1}{2})\pi)^{-1}$ and $y_n = ((2n+\frac{3}{2})\pi)^{-1}$. Then $f(x_n) = x_n$ and $f(y_n) = -y_n$, so

$$|x_n - y_n| = \frac{1}{(2n+\frac{1}{2})\pi} - \frac{1}{(2n+\frac{3}{2})\pi} = \frac{4}{(16n^2+16n+3)\pi};$$

$$|f(x_n) - f(y_n)| = \frac{1}{(2n+\frac{1}{2})\pi} + \frac{1}{(2n+\frac{3}{2})\pi} = \frac{4}{(16n^2+16n+3)\pi} = \frac{8(2n+1)}{(16n^2+16n+3)\pi}.$$

A small calculation shows that for $n = 109$, $|f(x_n) - f(y_n)| \approx \frac{1}{344.003} > \frac{1}{347}$, and $|x_n - y_n| \approx \frac{1}{150673}$. Thus the best value of $1/\delta$ is between 150673 and 481636.