Math 347 Homework 9 Solutions Due Wed., Nov. 7, 2001

1., 4. – 23.1 aceg, 24.5 (ungraded). The usual.

2. – 23.1 bdfh. In the first series, $a_n = n^{-n}$, hence $a_n^{1/n} = \frac{1}{n} \to 0$. Since $\beta = 0$, $R = \infty$ and $\sum \left(\frac{x}{n}\right)^n$ converges for all x.

In the second series, $a_n = \frac{n^3}{3^n}$, so that

$$a_n^{1/n} = \frac{n^{3/n}}{3} \to \frac{1}{3},$$

because $n^{3/n} = (n^{1/n})^3$. This means that $R = 3$. Observe that $|a_n x^n| = n^3 \geq 1$ when $x = \pm 3$, so the series does not converge at either endpoint and the interval of convergence is $(-3, 3)$.

In the third series, $a_n = \frac{1}{(n+1)^2 2^n}$, so

$$a_n^{1/n} = \frac{1}{(n+1)^{2/n} \cdot 2} \to \frac{1}{2},$$

since $(n+1)^{2/n} \to 1$. Again, we check the endpoints: for $|x| = 2$, $|a_n||x|^n = \frac{1}{(n+1)^2}$ and so the series converges at the endpoints and the interval of convergence is $[-2, 2]$.

Similarly, in the fourth series, $a_n = \frac{(-1)^n}{n^2 \cdot 4^n}$ and $|a_n|^{1/n} \to \frac{1}{4}$ and when $|x| = 4$, $|a_n x^n| = \frac{1}{n^2}$ and so the series converges at the endpoints and on $[-4, 4]$ overall. Interestingly, the only part of this problem in which alternating series play a role is (g).

3. – 23.4. (Note that $a_{2n} = (\frac{6}{5})^{2n}$ and $a_{2n+1} = (\frac{2}{5})^{2n+1}$.)

(a) It follows from the given information that if $b_n = |a_n|^{1/n}$, then $b_{2n} = \frac{6}{5}$ and $b_{2n+1} = \frac{2}{5}$. Thus, $\limsup b_n = \frac{6}{5}$ and $\liminf b_n = \frac{2}{5}$. Further,

$$\frac{a_{2n+1}}{a_{2n}} = \frac{(\frac{2}{5})^{2n+1}}{(\frac{6}{5})^{2n}} = \frac{2^{2n+1}}{5 \cdot 6^{2n}} = \frac{2}{5} \cdot \frac{1}{3^{2n}} \to 0.$$

and

$$\frac{a_{2n+2}}{a_{2n+1}} = \frac{(\frac{6}{5})^{2n+2}}{(\frac{2}{5})^{2n+1}} = \frac{6^{2n+2}}{5 \cdot 2^{2n+1}} = \frac{6}{5} \cdot 3^{2n+1} \to \infty.$$

Thus $\limsup \left|\frac{a_{n+1}}{a_n}\right| = \infty$ and $\liminf \left|\frac{a_{n+1}}{a_n}\right| = 0$. The ratio test is spectacularly useless in this case.

(b) Note that $|a_{2n}| > 1$ for $n \geq 1$; thus, neither $\sum a_n$ nor $\sum (-1)^n a_n$ converges.

(c) It follows from (a) that $\beta = \frac{6}{5}$, hence $R = \frac{5}{6}$, so the series converges at least on $(-\frac{5}{6}, \frac{5}{6})$. If $|x| = \frac{5}{6}$, then $a_{2n} x^{2n} = 1$, so $a_n x^n$ does not converge to 0 and the series does not converge at the endpoints.

9. It seems sensible to put the solution here; to find a "closed" form for the limit, assuming that we can rearrange the sum where it converges absolutely, i.e. on $(-\frac{5}{6}, \frac{5}{6})$. Using the suggestion, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\frac{6}{5}\right)^{2n}x^{2n}+\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^{2n+1}x^{2n+1} &= \sum_{n=0}^{\infty}\left(\frac{6^2x^2}{5^2}\right)^{n}+\frac{2x}{5}\sum_{n=0}^{\infty}\left(\frac{2^2x^2}{5^2}\right)^{n}\\ &= \frac{1}{1-\frac{36}{25}x^2}+\frac{2x/5}{1-\frac{4}{25}x^2} = \frac{25}{25-36x^2}+\frac{10x}{25-4x^2}\\ &= \frac{625+250x-100x^2-360x^3}{(25-36x^2)(25-4x^2)}.\end{aligned}$$

5. – 24.6. Let $f_n(x) = (x - \frac{1}{n})^2$ and $x \in [0,1]$. Let $f(x) = x^2$. Then $f_n(x) \to f(x)$ for every $x \in \mathbf{R}$ because $x - \frac{1}{n} \to x$. Furthermore

$$|f_n(x) - f(x)| = |x^2 - \tfrac{2}{n}x + \tfrac{1}{n^2} - x^2| \leq |\tfrac{2}{n}||x| + |\tfrac{1}{n^2}|.$$

If $0 \leq x \leq 1$, then this difference has an upper bound of $\frac{2}{n} + \frac{1}{n^2}$. Since this clearly $\to 0$ as $n \to \infty$, the convergence is uniform by 24.4.

6. – 24.8. Well, this looks familiar, doesn't it? We have

$$f_n(x) = \sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x},$$

and we know that this converges pointwise to $f(x) = \frac{1}{1-x}$ on the interval $[0,1)$ but not at $x = 1$, and (I hope) we know that the convergence is not uniform, because $|f_n(x)| \leq n+1$ and f is unbounded on $[0,1]$, hence $\sup\{|f_n(x) - f(x)|\} = \infty$ for all n.

7. – 24.10a. Routine epsilonics. If $f_n \to f$ and $g_n \to g$ on S, then given $\epsilon > 0$, there exists N_1 so that $n > N_1$ implies $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ for all $x \in S$ and there exists N_2 so that $n > N_2$ implies $|g_n(x) - g(x)| < \frac{\epsilon}{2}$ for all $x \in S$. Thus, if $n > \max\{N_1, N_2\}$, then

$$|f_n(x) + g_n(x) - (f(x) + g(x))| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

8. – 25.6. Actually very easy. Apply the Weierstrass M-Test with $M_k = |a_k|$. Then $\sum M_k$ converges by hypothesis and $M_k \geq |a_k x^k|$ for $x \in [-1,1]$. Thus, the series converges uniformly to *some* function, and since the partial sums are polynomials, they are continuous and 24.3 implies that the limit function must be continuous. For this reason, $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is a continuous function on $[-1,1]$.

Remark: There is no simple formula for f. However, as we shall soon see, power series can be differentiated and integrated term by term. Thus, it follows that

$$xf'(x) = x\sum_{n=1}^{\infty}\frac{nx^{n-1}}{n^2} = \sum_{n=1}^{\infty}\frac{x^n}{n} = -\log(1-x) \implies f(x) = \int_0^x \frac{\log(\frac{1}{1-t})}{t}\,dt.$$

10. – 24.14. Let

$$f_n(x) = \frac{nx}{1+n^2x^2}.$$

Then $f_n(0) = 0$, and if $x \neq 0$, then

$$|f_n(x)| = \frac{n|x|}{1+n^2x^2} < \frac{n|x|}{n^2x^2} = \frac{1}{n|x|}.$$

Since $x \neq 0$, $n|x| \to \infty$, hence $\lim f_n(x) = f(x) = 0$ for all x. But

$$f_n'(x) = \frac{-n(nx+1)(nx-1)}{(1+n^2x^2)^2},$$

hence $f_n'(x) > 0$ for $0 < x \leq \frac{1}{n}$ and $f_n'(x) < 0$ for $x > \frac{1}{n}$. That is, for $0 \leq x < \infty$,

$$f_n(x) \leq f_n\left(\frac{1}{n}\right) = \frac{1}{1+1} = \frac{1}{2}.$$

This means that

$$\sup\{|f_n(x) - f(x)| : x \in [0,1]\} = \frac{1}{2},$$

which does not converge to 0, so the convergence is not uniform on $[0,1]$. However,

$$\sup\{|f_n(x) - f(x)| : x \in [1,\infty]\} = f_n(1) = \frac{n}{1+n^2} \to 0,$$

hence the convergence is uniform for $[1,\infty]$.

What's going on here is that there is pointwise convergence everywhere and, essentially, uniform convergence on any interval that avoids a neighborhood of zero.